X. *On the Integration of certain differential Expressions, with which Problems in physical Astronomy are connected, &c.* By Robert Woodhouse, *A. M. F. R. S. Fellow of Caius College.*

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In analytical investigation, two important objects present themselves: the concise and unambiguous expression of the conditions of a problem in algebraic language; and the reduction of such expression into forms commodious for arithmetical computation.

If the introduction of the new calculi, as they have been called, has extended the bounds of science, it has enormously increased its difficulties, in their number and magnitude. The differential forms that can be completely integrated, occur in few problems only, and those of small moment. In physical astronomy, the investigations give rise to differential expressions, which call forth all the resources of the analytic art, even for their approximate integration.

For the integration of differential expressions that, by the process of taking the differential, can be derived from no finite algebraic form, recourse is had to infinite series: thus, if the expression be $dx \,.\, fx$, and there is no quantity $\mathrm{F}x$, such that $dx \,.\, fx = d\,(\mathrm{F}x)$: fx is put $= f\,((x-a)+a) = f\,(x-a) + \mathrm{D}f\,(x-a) \,.\, a + \underset{c}{\mathrm{D}}^2 f\,(x-a)\, a^2 +$ &c. and $\int dx \,.\, fx = \int dx \,.\, f\,(x-a) + \int dx \,.\, \mathrm{D}f\,(x-a) + \int dx \,.\, \underset{c}{\mathrm{D}}^2 f\,(x-a)$ &c. or,

putting $fx = f(a + x - a)$, the integral of $dx \,.\, fx$ is calculated from the series $\int dx \,.\, fa \,.\, (x - a) + \int dx \,.\, \mathrm{D}fa \,.\, (x - a)^2 + \&c.$

But, although the integrals of many expressions can thus be exhibited, yet such series are useless for the purpose of arithmetical computation, except their terms continually decrease, and except the limits of the ratio of the decrease of the terms can be determined; and the invention of series adapted to arithmetical computation, has not been the least of the difficulties encountered by modern analysts.

Although the differential expressions that admit no finite integration have not been reduced into classes, yet there are some, from their simplicity, and frequent occurrence in analytical investigation, more conspicuously known and attentively considered: such are the expressions $\frac{dx}{1+x}$, $\frac{dx}{\sqrt{(1-x^2)}}$; and the computation of their integrals, in other words, is the determination of the logarithms of numbers, and the lengths of circular arcs.

The necessity of calculating the integrals of expressions such as $\frac{dx}{1+x}$, $\frac{dx}{\sqrt{(1-x^2)}}$, must soon have obtruded itself on the attention of the early analysts: for several expressions, as $\frac{dx}{1-x^2}$, $\frac{dx}{x\sqrt{1-x^2}}$, $\frac{dx}{1+x^2}$, $\frac{dx}{x\sqrt{x^2-1}}$, &c. apparently dissimilar, are easily reduced to the forms $\frac{dx}{1+x}$, $\frac{dx}{\sqrt{1-x^2}}$; and besides, the difficulty of integrating a variety of forms, is soon reduced to that of the integration of $\frac{dx}{1+x}$, $\frac{dx}{\sqrt{1-x^2}}$: such, for instance, are the forms $\frac{dx}{x\sqrt{(1-x^2)}}$, $\frac{dx}{x^3\sqrt{(1-x^2)}}$, and all that are comprehended under $\frac{dx}{x^{2m+1}\sqrt{(1-x^2)}}$; the forms $\frac{x^2\,dx}{\sqrt{(1-x^2)}}$, $\frac{x^4\,dx}{\sqrt{(1-x^2)}}$, &c. and all that are comprehended under $\frac{x^{2m}\,.\,dx}{\sqrt{(1-x^2)}}$.

It is on the grounds of convenience of calculation, and of systematic arrangement, that differential expressions, such as have been just exhibited, are resolved into a series of terms $Pdx + P'dx + P''dx + \&c. + Q\frac{dx}{\sqrt{1 \pm x^2}}$, where $Pdx + P'dx$, $P''dx$ are integrable; for, remove those grounds, and it will be difficult to assign a reason why * $\frac{x^{2m+1}}{2m+1} + \frac{x^{2m+3}}{2m+3} \cdot \frac{1}{2} + \frac{x^{2m+5}}{2m+5} \cdot \frac{1 \cdot 3}{2 \cdot 4} + \&c.$ is not an integral of $\frac{x^{2m} \cdot dx}{\sqrt{(1-x^2)}}$ equally exact as $-\sqrt{(1-x^2)}\left\{\frac{x^{2m-1}}{2m} + \frac{1 \cdot (2m-1)}{(2m-2) \cdot 2m} \cdot x^{2m-3} + \frac{1 \cdot (2m-1)(2m-3)}{(2m-4)(2m-2) \cdot 2m} \cdot x^{2m-5} + \&c.\right\} + \frac{1 \cdot 3 \cdot 5 \ldots \cdot 2m-1}{2 \cdot 4 \cdot 6 \ldots \cdot 2m} \int \frac{dx}{\sqrt{(1-x^2)}}$.

In the application of the differential calculus to curve lines, after making certain arbitrary assumptions, it appears that hyperbolic areas, and arcs of circles, may be computed from the integrals of the expressions $\frac{dx}{1+x}$, $\frac{dx}{\sqrt{(1-x^2)}}$; the integrals of which are in fact afforded by the several methods that relate to the quadratures of the circle and hyperbola; and mathematicians, either for the sake of embodying in some degree their speculations, or from a notion of a necessary connexion subsisting between circles, hyperbolas, and the integrals of $\frac{dx}{1+x}$, $\frac{dx}{\sqrt{(1-x^2)}}$, have expressed the integrals by the arcs and areas of those figures. Although the computation of the integrals, is totally independent of the existence of the figures, and of their properties, yet it is curious, that the simplest transcendental expressions of analysis, should express parts of the simplest figures in geometry.

* This series arises from expanding $\frac{1}{\sqrt{1-x^2}}$, and from integrating each term multiplied into $x^{2m} \cdot dx$.

In analytical investigation, after $\frac{dx}{1+x}$, $\frac{dx}{\sqrt{(1-x^2)}}$, the transcendental expression, next, in point of simplicity, is $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$;* in a particular application, this differential represents the arc of an ellipse,† a figure, next, in point of simplicity, to the circle.

Many differential expressions depending, for their integration, on the integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, it became necessary to exhibit it, for all values of x and e. A problem in consequence arose, of no small difficulty, named, analogously to the naming of $\int\frac{dx}{\sqrt{1-x^2}}$, the rectification of the ellipse. In the prosecution of the researches to which this problem led, it was discovered that the hyperbola might be rectified by means of the ellipse, or, to speak correctly, and without the employment of figurative language, it was discovered that the transcendental expression $dx\sqrt{\left(\frac{e^2x^2-1}{x^2-1}\right)}$ $(e > 1)$ might be made to depend, for its integration, on that of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ $e < 1$.

The integration of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ does not depend more on the length of an ellipse, than it does, on the time of the vibration of a pendulum in a circular arc, or on the attraction of a spheroid; but, in each of these problems, it occurs as an analytical phrase, an expression in symbolical language, the exact meaning of which it is necessary to know. If the meaning be determined for one case, it is for all three; and hence, with the rectification of

* $\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$ is as simple an expression: they are considered together in the following pages.

† The ellipse admits of an easy mechanical description; and, considered as a section of the cone, was admitted by the ancient geometricians into plane geometry.

the ellipse, a problem by itself unimportant, the solutions of other problems, are intimately connected; and, with this object in view, the determination of the length of a curve line, mathematicians have enriched analysis with several curious artifices, and valuable methods.

To determine the integrals of $\frac{dx}{\sqrt{1-x^2}}$, $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, it is necessary to expand them into series. The difficulty is, to expand them into series that converge: the determination of the integral of $\frac{dx}{\sqrt{(1-x^2)}}$ ought to precede that of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$; indeed, in most of the series that represent the latter, $\int\frac{dx}{\sqrt{(1-x^2)}}$ is involved as a term, and is supposed to be known. The determination of each integral presents a curious circumstance, in the correspondence of certain geometrical properties and analytical artifices; for instance, the theorem for the tangent of the sum of two circular arcs, affords, analytically, a means of computing the length of the arc; and, conversely, the analytical artifice* by which the integral of $\frac{dx}{\sqrt{(1-x^2)}}$ is computed; translated, leads to the

* The method of deducing the value of $\int\frac{dx}{\sqrt{1-x^2}}$ between the values of x, 0 and 1, independently of any reference to a circle, is as follows.

Let $\frac{dx}{\sqrt{1-x^2}}=\frac{du'}{\sqrt{1-u'^2}}+\frac{du''}{\sqrt{1-u''^2}}$ then $\int\frac{dx}{\sqrt{(1-x^2)}}=\int\frac{du'}{\sqrt{(1-u'^2)}}+\int\frac{du''}{\sqrt{(1-u''^2)}}$ $+C$, and, expressing the integrals by their exponential expressions, we may deduce (see Phil. Trans. 1802) $u'\sqrt{(1-u''^2)}+u''\sqrt{(1-u'^2)}=x$. Let $x=1$ and $u'=u''$ $=\therefore\frac{1}{\sqrt{2}}$, consequently $\frac{2du'}{\sqrt{(1-u'^2)}}=\frac{dx}{\sqrt{(1-x^2)}}$, or twice the integral of $\frac{du'}{\sqrt{(1-u'^2)}}$ between the values of u', 0 and $\frac{1}{\sqrt{2}}$, equals the integral of $\frac{dx}{\sqrt{(1-x^2)}}$ between the values of x, 0 and 1. Again, put $\frac{du'}{\sqrt{(1-u'^2)}}=\frac{dv}{\sqrt{(1-v^2)}}+\frac{dv'}{\sqrt{(1-v'^2)}}$ $\therefore$ as before, $v\sqrt{(1-v'^2)}+v'\sqrt{(1-v^2)}=u'$; put $u'=\frac{1}{\sqrt{2}}$, $v=\frac{1}{\sqrt{5}}$ and $v'=\frac{1}{\sqrt{10}}$, consequently $\int\frac{dx}{\sqrt{(1-x^2)}}=2\int\frac{dv'}{\sqrt{(1-v'^2)}}$ (contained between the values of v', 0 and $\frac{1}{\sqrt{10}}$) +

properties of the sines, and tangents, of circular arcs. Again, FAGNANI's theorem, by which a right line is assigned equal to the difference of two elliptic arcs, affords a method of arithmetically computing the length of the ellipse; and, conversely, the analytical artifice by which the integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ is computed; translated into geometrical language, becomes FAGNANI's theorem. And again, the analytical resolution of $\int dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ into $Au' + Bu'' + P\int du'\sqrt{\left(\frac{1-e'^2u'^2}{1-u'^2}\right)} + Q\int du''\sqrt{\left(\frac{1-e''^2u''^2}{1-u''^2}\right)}$, (where the integrals, on account of the

$2\int\frac{dv}{\sqrt{(1-v^2)}}$ (contained between the values of v, o and $\frac{1}{\sqrt{5}}$), which latter series, from the smallness of v, v', converge with considerable rapidity; or the latter part thus, put $v = \frac{z}{\sqrt{(1+z^2)}}$, $v' = \frac{z'}{\sqrt{(1+z'^2)}}$, $u' = \frac{y}{\sqrt{(1+y^2)}}$, then $\frac{z+z'}{1-zz'} = y$, and $\int\frac{dy}{1+y^2}$ $= \int\frac{dz}{1+z^2} + \int\frac{dz'}{1+z'^2}$.

Now, if $u' = \frac{1}{\sqrt{2}}$, $y = 1$,

if $v = \frac{1}{\sqrt{5}}$, $z = \frac{1}{2}$,

if $v' = \frac{1}{\sqrt{10}}$, $z' = \frac{1}{3}$.

Consequently, the integral of $\frac{dy}{1+y^2}$ (between the values of y, o and 1) $= \int\frac{dz}{1+z^2}$ (between the values of z, o and $\frac{1}{2}$) $+ \int\frac{dz'}{1+z'^2}$ (between the values of z', o and $\frac{1}{3}$), and consequently, $\int\frac{dx}{\sqrt{(1-x^2)}}$ (between o and 1) $= 2\left\{\frac{1}{2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \&c.\right\}$ $+ 2\left\{\frac{1}{3} - \frac{2}{3.3^3} + \frac{1}{5.3^5} - \&c.\right\}$ which is, in fact, EULER's method of determining the periphery of a circle. Now, from this analytical artifice of putting the integral of $\frac{dx}{\sqrt{(1-x^2)}} = \int\frac{d'u'}{\sqrt{(1-u'^2)}} + \int\frac{du''}{\sqrt{(1-u''^2)}}$, by which means its arithmetical value is computed, may be deduced those theorems which relate to the sines, and tangents, of the sum and difference of arcs, &c. by translating the formula $u'\sqrt{(1-u''^2)} + u''\sqrt{(1-u'^2)} = x$ into geometrical language.

smallness of e', e'', are readily computed,) translated into the language of geometry, expresses a curious relation between the arcs of three ellipses, the excentricities of which vary according to a certain law.

Hence it appears, that there are two different methods by which the analytic art may be advanced; either by artifices peculiarly its own, or by aid drawn from the properties of figures and curve lines; if, for instance, FAGNANI's theorem be proved for an ellipse, by processes purely geometrical, then, such a theorem, expressed in analytical language, becomes immediately a means of computing the integral of $dx\sqrt{\frac{1-e^2x^2}{1-x^2}}$; or if, by reasonings strictly geometrical, a relation can be established between the arcs of three ellipses, whose excentricities vary according to a certain law, then, by expressing such a relation in the signs of algebra, the integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ may be computed by means of the integrals of $du'\sqrt{\left(\frac{1-e'^2u'^2}{1-u'^2}\right)}$, and of $du''\sqrt{\left(\frac{1-e''^2u''^2}{1-u''^2}\right)}$; which integrals can be found more readily than the original integral, by reason of the quicker convergency of the series into which the differential expressions may be expanded, e' and e'' being less than e.

One main object of the present paper is, to exhibit the integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ for all values of e, and to reduce other integrals to it. Much has been already done on this subject. The researches of mathematicians on the length and comparison of elliptic arcs, are extended over the surface of many memoirs; yet I hope to have something to add in point of invention, and more in point of arrangement and simplicity of expression. The labours of future students will surely be lessened, if it be

shown, that several methods, apparently distinct and dissimilar, because expressed in different language, are fundamentally, and in principle, the same.

The simplest mode, and the first that occurred to mathematicians, of finding the value of $\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ was, to expand the differential expression into a series of terms ascending by the powers of e, and to take the integral of each term. This method, however, is very imperfect; for, if e be nearly $= 1$, the series converges so slowly as to be unfit, or at least very incommodious, for arithmetical computation. It became necessary then to possess a series ascending by the powers of $1 - e^2$; and such a series was first given by EULER, in his Opuscula, published at Berlin in 1750; and it must be manifest, that there can be no one single series, ascending by the powers of e, or by powers of the same function e, that can in all cases represent its value. I purpose to consider the several series that represent the value of $\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$,

when e is small,

when e is nearly $= 1$, or, when $\sqrt{(1-e^2)}$ is small,

when e is $\angle \sqrt{(1-e^2)}$ and $\angle \frac{1}{\sqrt{2}}$,

when e is $> \sqrt{(1-e^2)}$ and $> \frac{1}{\sqrt{2}}$,

when e and $\sqrt{(1-e^2)}$ are equal, or when each equals $\frac{1}{\sqrt{2}}$.

The series for the first and second cases, I shall deduce, because I wish to consider the subject in its fullest extent; but those series, when we regard practical commodiousness, are superseded by the methods by which the $\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ is to be found, in the third and fourth cases. Two methods then, are only requisite for finding the integral in all the values of e; for the integral

in the last case may be found, with nearly equal convenience, by either of the methods in the two preceding cases.

For the sake of conciseness, I employ the symbol D to denote the numeral coefficients of the terms arising from the expansion of $(1-x)^m$; thus, $\mathrm{D}1^m$ signifies m; $\mathrm{D}^2\, 1^m$, $m \,.\, m-1$; $\underset{c}{\mathrm{D}}^2\, 1^m$, $\frac{m\,.\,m-1}{1\,.\,2}$; $\underset{c}{\mathrm{D}}^3\, 1^m$, $\frac{m\,.\,(m-1)\;(m-2)}{1\,.\,2\,.\;\;3}$; $\underset{c}{\mathrm{D}}^n\, 1^m$ signifies, $\frac{m\,.\,(m-1)\,(m-2)\,..\,m-n+1}{1\,.\,2\,.\;\;3\;\ldots\;n}$, and consequently, in particular values of m and n, $\underset{c}{\mathrm{D}}^3\, 1\tfrac{1}{2}$ signifies $\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}$; $\underset{c}{\mathrm{D}}^4\, 1\tfrac{1}{2}$ signifies $-\frac{1\,.\,1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}$; $\underset{c}{\mathrm{D}}^3\, 1^{-\frac{1}{2}}$ signifies $-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}$; $\underset{c}{\mathrm{D}}^4\, 1^{-\frac{1}{2}}$, signifies $\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}$; &c.

Employing, therefore, this notation in the expansion of $\sqrt{(1-e^2x^2)}$, we have $dx\sqrt{\left(\frac{1-e^2\,x^2}{1-x^2}\right)} = \frac{dx}{\sqrt{1-x^2}}\Big\{1\tfrac{1}{2} - \mathrm{D}1\tfrac{1}{2}\, e^2\, x^2 + \underset{c}{\mathrm{D}}^2\, 1\tfrac{1}{2}\, e^4\, x^4 - \underset{c}{\mathrm{D}}^3\, 1\tfrac{1}{2}\, e^6\, x^6 + \text{\&c.}\Big\}$, and the $(n+1)$th term is

$$\underset{c}{\mathrm{D}}^n\, 1\tfrac{1}{2}\, e^{2n} \,.\, \frac{x^{2n}\,.\,dx}{\sqrt{(1-x^2)}}.$$

Now, $d\left(x^{2n-1}\sqrt{(1-x^2)}\right) = (2n-1)\,\frac{x^{2n-2}\,dx}{\sqrt{(1-x^2)}} - \frac{2n\,x^{2n}.\,dx}{\sqrt{(1-x^2)}}$; hence,

$$\int\frac{x^{2n}\,dx}{\sqrt{(1-x^2)}} = -\frac{1}{2n}\,x^{2n-1}\sqrt{(1-x^2)} + \frac{2n-1}{2n}\int\frac{x^{2n-2}\,dx}{\sqrt{(1-x^2)}}$$

$$= -\frac{1}{2n}\,x^{2n-1}\sqrt{(1-x^2)} - \frac{2n-1}{2n\,.\,2n-2}\,x^{2n-3}\sqrt{(1-x^2)}$$

$$+\frac{(2n-1)\,.\,(2n-3)}{2n\,.\,(2n-2)}\,.\int\frac{x^{2n-4}\,dx}{\sqrt{(1-x^2)}};$$

consequently, continuing the reduction,

$$\int\frac{x^{2n}\,dx}{\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)}\Big\{\frac{x^{2n-1}}{2n} + \frac{(2n-1)}{2n\,.\,(2n-2)}\,.\,x^{2n-3} + \text{\&c.}\Big\}$$

$$+\frac{(2n-1)\,.\,(2n-3)\;\text{\&c.}\;\ldots\ldots\,5\,.\,3\,.\,1}{2n\;.\;2n-2\;\ldots\ldots\ldots\,6\,.\,4\,.\,2}\int\frac{dx}{\sqrt{(1-x^2)}}\;(\varphi).$$

Hence, putting for n the several values 0, 1, 2, 3, &c. we have

$$\int dx\sqrt{\left(\frac{1-e^2\,x^2}{1-x^2}\right)} = \varphi$$

$$-\,\mathrm{D}1\tfrac{1}{2}\,e^2\Big\{\frac{-x\sqrt{(1-x^2)}}{2} + \frac{\varphi}{2}\Big\}$$

$$+\,\underset{c}{\mathrm{D}}^2\,1\tfrac{1}{2}\,e^4\Big\{\frac{-x^3\sqrt{(1-x^2)}}{4} - \frac{3x\sqrt{(1-x^2)}}{4\,.\,2} + \frac{1\,.\,3}{2\,.\,4}\,\varphi\Big\}$$

$- \underset{c}{\mathrm{D}}^3\, 1^{\frac{1}{2}} e^6 \left\{ \frac{-x^5\sqrt{(1-x^2)}}{6} - \frac{5x^3\sqrt{(1-x^2)}}{6\,.\,4} - \frac{5\,.\,3\,.\,x\,\sqrt{(1-x^2)}}{6\,.\,4\,.\,2} + \frac{1\,.\,3\,.\,5\,.\,\varphi}{2\,.\,4\,.\,6} \right\}$
$+$ &c.

Hence, if the integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, between the values of x, o and 1, be required, putting $\frac{\pi}{2} = \frac{3\,.\,14159}{2} =$ value of φ, or of $\int \frac{dx}{\sqrt{(1-x^2)}}$, when $x = 1$, we have

$\int dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ (from $x=$ o to $x = 1$)

$$= \frac{\pi}{2}\left\{1 - \mathrm{D}1^{\frac{1}{2}}.e^2.\tfrac{1}{2} + \underset{c}{\mathrm{D}}^2\, 1^{\frac{1}{2}}\,.\,e^4\,.\,\frac{1\,.\,3}{2\,.\,4} - \underset{c}{\mathrm{D}}^3\, 1^{\frac{1}{2}} e^6\,.\,\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6} + \text{\&c.}\right\} (1)$$

or, developing the symbolical coefficients $\mathrm{D}1^{\frac{1}{2}}$, $\mathrm{D}^2\,1^{\frac{1}{2}}$, &c.

$$= \frac{\pi}{2}\left\{1 - \frac{1\,.\,1}{2\,.\,2}e^2 - \frac{1\,.\,1}{2\,.\,4}\cdot\frac{1\,.\,3}{2\,.\,4}e^4 - \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\cdot\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}e^6 - \text{\&c.}\right\}$$

which series has been given by several authors, SIMPSON, EULER, *Animadversiones in Rect. Ellips.* p. 129, &c.

If, instead of the coefficients $\frac{1}{2}\cdot\frac{1\,.\,3}{2\,.\,4}$, we use $-\mathrm{D}1^{-\frac{1}{2}}, \underset{c}{\mathrm{D}}^2\, 1^{-\frac{1}{2}}$, &c. the integral

$$= \frac{\pi}{2}\left\{1 + \mathrm{D}1^{\frac{1}{2}}\,.\,\mathrm{D}1^{-\frac{1}{2}}\,.\,e^2 + \underset{c}{\mathrm{D}}^2\, 1^{\frac{1}{2}}\,.\,\underset{c}{\mathrm{D}}^2\, 1^{-\frac{1}{2}}\,.\,e^4 + \underset{c}{\mathrm{D}}^3\, 1^{\frac{1}{2}}\,.\,\underset{c}{\mathrm{D}}^3\, 1^{-\frac{1}{2}}\,.\,e^6 + \text{\&c.}\right\}$$

where the $(n+1)$th term is $\underset{c}{\mathrm{D}}^n\, 1^{\frac{1}{2}}\,.\,\underset{c}{\mathrm{D}}^n\, 1^{-\frac{1}{2}}$, which, (since $\underset{c}{\mathrm{D}}^n\, 1^{-\frac{1}{2}} = -\underset{c}{\mathrm{D}}^n\, 1^{\frac{1}{2}}.\,(2n-1)$), equals $-\left(\underset{c}{\mathrm{D}}^n\, 1^{\frac{1}{2}}\right)^2.\,(2n-1)$; consequently, the integral may be put

$$\frac{\pi}{2}\left\{1 - (\mathrm{D}1^{\frac{1}{2}})^2\,.\,e^2 - \left(\underset{c}{\mathrm{D}}^2\, 1^{\frac{1}{2}}\right)^2.\,3e^4 - \left(\underset{c}{\mathrm{D}}^3\, 1^{\frac{1}{2}}\right)^3.\,5e^6 - \text{\&c.}\right\}$$

From this series, $\int dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ may be computed when e is small; but it is evidently of very little use when e is either nearly $= 1$, or is of mean value. To speak in geometrical language, the length of an ellipse of small excentricity may be computed by the above series.

If v be put $= 1 - 2x^2$, $\frac{dx}{\sqrt{(1-x^2)}} = \frac{-dv}{2\sqrt{(1-v^2)}}$,
and $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)} = \frac{-dv}{2\sqrt{(1-v^2)}}\cdot\sqrt{\frac{(2-e^2)}{2}}\sqrt{\left(1+\frac{e^2v}{2-e^2}\right)}$,

put $\frac{e^2}{2-e^2}=c$, and $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}=df$,

then $df=\frac{1}{2}\sqrt{\frac{e^2}{2c}}\cdot\frac{dv}{\sqrt{(1-v^2)}}\left\{1^{\frac{1}{2}}+D\,1^{\frac{1}{2}}.cv+\frac{D^2}{c}\,1^{\frac{1}{2}}.c^2v^2+\text{\&c.}\right\}$

Now (by methods similar to those that have been given)

$$\int\frac{v^{2n}\,dv}{\sqrt{(1-v^2)}}=-\sqrt{(1-v^2)}\left\{\frac{v^{2n-1}}{2n}+\frac{2n-1\,v^{2n-3}}{2n\,.\,(2n-2)}+\text{\&c.}\right\}+\frac{1\cdot 3\ldots.2n-1}{2\cdot 4\ldots2n}$$

$$\int\frac{dv}{\sqrt{1-v^2}}\text{, and}\int\frac{v^{2n+1}\,dv}{\sqrt{(1-v^2)}}=-\sqrt{1-v^2)}\left\{\frac{v^{2n}}{2n+1}+\frac{1\,.\,2n\,v^{2n-2}}{(2n+1)\,(2n-1)}\right.\cdot$$

$$\left.+\text{\&c.}+\frac{1\cdot 2\cdot 4\ldots.2n}{1\cdot 3\cdot 5\ldots.2n+1}\right\}.$$

Hence, putting $\varphi'=\int\frac{-dv}{\sqrt{(1-v^2)}}$

$$f=\frac{1}{2}\sqrt{\frac{e^2}{2c}}\left\{\begin{array}{l}\varphi'+D\,1^{\frac{1}{2}}c\sqrt{(1-v^2)}+\frac{D^2}{c}\,1^{\frac{1}{2}}c^2\left\{\frac{v\sqrt{(1-v^2)}}{2}+\frac{\varphi'}{2}\right\}\\+\frac{D^3}{c}\,1^{\frac{1}{2}}.c^3\left\{.\frac{v^2}{3}\sqrt{(1-v^2)}+\frac{1\,.\,2}{2\,.\,4}\right\}+\text{\&c.}\end{array}\right\}$$

which series agrees exactly with LEGENDRE's, given in *Mem. de l'Acad.* p. 620, when the quantities v, $\sqrt{1-v^2}$, &c. are expressed in geometrical language.

In order to find the integral from $x=0$ to $x=1$, put $x=0$, then $v=1$, put $x=\frac{1}{\sqrt{2}}$, and then $v=0$; but it has appeared that the $\int\frac{dx}{\sqrt{(1-x^2)}}$ (between the values of x, 0 and 1) $=2\int\frac{dx}{\sqrt{(1-x^2)}}$ (between the values of x, 0 and $\frac{1}{\sqrt{2}}$) $=$ consequently, $2\int\frac{dv}{\sqrt{(1-v^2)}}$ (between the values of v, 1 and 0).

Hence, $f=\sqrt{\left(\frac{2-e^2}{2}\right)}\frac{\pi}{2}\left\{1+\frac{D^2}{c}\,1^{\frac{1}{2}}.\frac{c^2}{2}+\frac{D^4}{c}\,1^{\frac{1}{2}}.\frac{1\cdot 3}{2\cdot 4}.c^4+\text{\&c.}\right\}$ (2)

or $=\sqrt{\left(1-\frac{e^2}{2}\right)}\frac{\pi}{2}\left\{1-\frac{1\cdot 1}{2\cdot 4}\cdot\frac{1}{2}\cdot c^2-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1\cdot 3}{2\cdot 4}c^4-\text{\&c.}\right\}$

which is the series given by LEGENDRE, and by EULER, *Novi Comm. Petrop.* Tom. XVIII. p. 71, and called by that author *Series maxume convergens;* yet the series is by no means practically commodious when e is nearly 1.

A very useful series, when e is small, was given by Mr. IVORY,

in the Edinburgh Transactions, Vol. IV. which I shall notice in the sequel; not now, because I consider it as a particular case of the general method by which, in all cases, the integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ may be computed.

In order to deduce the series by which, when e is nearly $=1$, f may be computed, put $1-e^2=b^2$,

then $df = dx\sqrt{\left(1+\frac{(1-e^2)x^2}{1-x^2}\right)} = dx\sqrt{\left(1+\frac{b^2x^2}{1-x^2}\right)} = \left(\text{if } x = \frac{z}{\sqrt{(1+z^2)}}\right)$

$$\frac{dz}{(1+z^2)^{\frac{3}{2}}}\sqrt{(1+b^2z^2)} = d\left\{\frac{z}{\sqrt{1+z^2}}\sqrt{(1+b^2z^2)}\right\} - \frac{b^2z^2\,dz}{\sqrt{1+z^2}\sqrt{(1+b^2z^2)}}.$$

Now, $\frac{b^2z^2\,dz}{\sqrt{((1+z^2)(1+b^2z^2))}} = \frac{b^2z^2\,dz}{\sqrt{(1+z^2)}}\left\{1^{-\frac{1}{2}} + D1^{-\frac{1}{2}}b^2z^2 + \underset{c}{D}^2 1^{-\frac{1}{2}}b^4z^4 + \&c.\right\}$

and the $(n+1)$th term $= \underset{c}{D}^n 1^{-\frac{1}{2}} b^{2n+2}.\frac{z^{2n+2}.\,dz}{\sqrt{(1+z^2)}}$.

Now, $\int\frac{z^{2n+2}.\,dz}{\sqrt{(1+z^2)}} = \frac{z^{2n+1}\sqrt{(1+z^2)}}{2n+2} - \frac{2n+1}{2n+2}\int\frac{z^{2n}\,dz}{\sqrt{(1+z^2)}}$; and, consequently,

$$= \frac{z^{2n+1}\sqrt{(1+z^2)}}{2n+2} - \frac{2n+1}{2n+2\,.\,2n}.z^{2n-1}\sqrt{(1+z^2)} + \frac{(2n+1)(2n-1)}{(2n+2)\,2n}\int\frac{z^{2n-2}\,dz}{\sqrt{1+z^2}}$$

$$= \sqrt{(1+z^2)}\left\{\frac{z^{2n+1}}{2n+2} - \frac{2n+1\,.\,z^{2n-1}}{2n+2\,.\,2n} + \frac{2n+1)(2n-1)}{(2n+2)\,2n\,.\,(2n-2)}.z^{2n-3} + \&c.\right\}$$

$$\pm\frac{(2n+1)(2n-1)\,.\,.\,.\,.\,.\,3\,.\,1}{(2n+2)\,2n\,.\,\,.\,.\,.\,.\,.\,4\,.\,2}.\int\frac{dz}{1+z^2}\left(\underset{c}{D}^{n+1}1^{-\frac{1}{2}}.\int\frac{dz}{\sqrt{(1+z^2)}}\right).$$

Hence, $\int\frac{dz}{(1+z^2)^{\frac{3}{2}}}\sqrt{(1+b^2z^2)}$

$= \frac{z}{\sqrt{(1+z^2)}}\sqrt{(1+b^2z^2)}$

$-\left\{D1^{-\frac{1}{2}}b^2 + D1^{-\frac{1}{2}}\underset{c}{D}^2 1^{-\frac{1}{2}}.b^4 + \underset{c}{D}^2 1^{-\frac{1}{2}}\underset{c}{D}^3 1^{-\frac{1}{2}}b^6 + \underset{c}{D}^3 1^{-\frac{1}{2}}\underset{c}{D}^4, 1^{-\frac{1}{2}}.b^8 + \&c.\right\}$

$\times$ log. $(z+\sqrt{(1+z^2)})$

$-b^2.\frac{z\sqrt{(1+z^2)}}{2}$

$-D1^{-\frac{1}{2}}b^4\sqrt{(1+z^2)}\left\{\frac{z^3}{4} - \frac{3z}{4\,.\,2}\right\}$

$-\underset{c}{D}^2 1^{-\frac{1}{2}}b^6\sqrt{(1+z^2)}\left\{\frac{z^5}{6} - \frac{5z^3}{6\,.\,4} + \frac{5\,.\,3\,.\,z}{6\,.\,4\,.\,2} - \&c.\right\}$

$-$ &c.

or, since $\sqrt{(1+z^2)} = \frac{1+z^2}{\sqrt{(1+z^2)}}$

$$\int\frac{dz}{\sqrt{(1+z^2)}}\sqrt{(1+b^2z^2)}$$
$$=\frac{z\sqrt{(1+b^2z^2)}}{\sqrt{(1+z^2)}}$$
$$-\left\{D1^{-\frac{1}{2}}b^2+D1^{-\frac{1}{2}}\underset{c}{D}^21^{-\frac{1}{2}}b^4+\underset{c}{D}^21^{-\frac{1}{2}}.\underset{c}{D}^31^{-\frac{1}{2}}\right\}\left\{l.z+\sqrt{(1+z^2)}-\frac{z}{\sqrt{(1+z^2)}}\right\}$$
$$-\frac{b^2z}{\sqrt{1+z^2}}\left\{\frac{z^2}{2}\right\}$$
$$-D1^{-\frac{1}{2}}\frac{b^4.z}{\sqrt{(1+z^2)}}\left\{\frac{z^4}{4}-\frac{z^2}{4.2}\right\}$$
$$-\underset{c}{D}^21^{-\frac{1}{2}}.\frac{b^6.z}{\sqrt{(1+z^2)}}\left\{\frac{z^6}{6}-\frac{z^4}{6.4}+\frac{z^2}{6.4.2}\right\}$$
$$-\ \&c.$$

Now, from this series, as it stands, the whole integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ cannot be computed, because x being $=\frac{z}{\sqrt{(1+z^2)}}$ when $x=1$, z is infinite: therefore, we must use an artifice similar to that by which $\int\frac{dx}{\sqrt{(1-x^2)}}$ has been computed; which artifice consists in finding v a function of x, such that $\int\frac{dx}{\sqrt{1-x^2}}$ (between $x=0$ and $x=a$, $a\angle 1$) $+\int\frac{dv}{\sqrt{1-v^2}}$ shall $=$ whole integral of $\frac{dx}{\sqrt{(1-x^2)}}\{$from $x=0$ to $x=1.\}$

Let therefore $x=\sqrt{\left(\frac{1-v^2}{1-e^2v^2}\right)}$, in which case, $\frac{x\sqrt{(1-x^2)}}{\sqrt{(1-e^2x^2)}}=\frac{v\sqrt{(1-v^2)}}{\sqrt{(1-e^2v^2)}}$; consequently, $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}=-\frac{dv(1-e^2)}{\sqrt{(1-v^2)}(1-e^2v^2)^{\frac{3}{2}}}$,

and $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}+dv\sqrt{\left(\frac{1-e^2v^2}{1-v^2}\right)}=e^2dv\frac{(1-2v^2+e^2v^4)}{\sqrt{(1-v^2)}(1-e^2v^2)^{\frac{3}{2}}}$
$=e^2\times d\left\{\frac{v\sqrt{(1-v^2)}}{\sqrt{(1-e^2v^2)}}\right\}$.

Hence,

$$\int dx\sqrt{\left(\frac{(1-e^2x^2)}{(1-x^2)}\right)}(f)+\int dv\sqrt{\left(\frac{1-e^2v^2}{1-v^2}\right)}=\frac{e^2.v\sqrt{(1-v^2)}}{\sqrt{(1-e^2v^2)}}+\text{Corr (C)}$$

when $x=1$, $v=0$. Let the whole integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, from $x=0$ to $x=1$, be denoted by $f(1)\therefore C=f(1)$,

and $f + \int dv \sqrt{\left(\frac{1-e^2 v^2}{1-v^2}\right)} = f(1) + \frac{e^2 . v\sqrt{(1-v^2)}}{\sqrt{(1-e^2 v^2)}}$.

Now, $\frac{e^2 v\sqrt{(1-v^2)}}{\sqrt{(1-e^2 v^2)}} = 0$, both when v is 0 and when v is 1; consequently, there is an intermediate value of v, with which $\frac{e^2 v\sqrt{(1-v^2)}}{\sqrt{(1-e^2 v^2)}}$ is a maximum. Such value of v, investigated, appears to be $\frac{1}{\sqrt{(1+\sqrt{(1-e^2)})}}$ = also x; consequently, $2f = f(1) + 1 - \sqrt{(1-e^2)}$ $= f(1) + 1 - b$.*

Now, from this property of the integral of $dx\sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, may the whole integral be computed; for, since $x = \frac{1}{\sqrt{(1+b)}}$, $z = \frac{1}{\sqrt{b}}$; consequently, $f(1) = 2f - 1 + b$

$$
\begin{aligned}
&= 1 + b \quad \left(\text{for, putting } z = \frac{1}{\sqrt{b}},\ z\sqrt{\left(\frac{1+b^2 z^2}{1+z^2}\right)} = 1\right) \\
&- 2\left\{ \mathrm{D}1^{-\frac{1}{2}} b^2 + \mathrm{D}1^{-\frac{1}{2}} . \underset{c}{\mathrm{D}}^2 1^{-\frac{1}{2}} b^4 + \underset{c}{\mathrm{D}}^2 1^{-\frac{1}{2}} . \underset{c}{\mathrm{D}}^3 1^{-\frac{1}{2}} b^6 + \&c. \right\} \\
&\qquad \left\{ l . \frac{1+\sqrt{(1+b)}}{\sqrt{b}} - \frac{1}{\sqrt{(1+b)}} \right\} \\
&- \frac{2}{\sqrt{(1+b)}} . \frac{b}{2} \\
&- \frac{2\mathrm{D}1^{-\frac{1}{2}}}{\sqrt{(1+b)}} \left\{ \frac{b^2}{4} - \frac{b^3}{4 . 2} \right\} \qquad\qquad (3) \\
&- \frac{2\underset{c}{\mathrm{D}}^2 1^{-\frac{1}{2}}}{\sqrt{(1+b)}} \left\{ \frac{b^3}{6} - \frac{b^4}{6 . 4} + \frac{b^5}{6 . 4 . 2} \right\}, \\
&- \&c.
\end{aligned}
$$

This form is, in fact, the same as what is given by LEGENDRE, *Mem. de l'Acad.* 1786; and, if the integral had been taken by a method a little different from the above, a series exactly coinciding with LEGENDRE's would have resulted. Thus,

since $df = dz \frac{\sqrt{(1+b^2 z^2)}}{(1+z^2)^{\frac{3}{2}}} = \frac{dz}{(1+z^2)^{\frac{3}{2}}} \left\{ 1^{\frac{1}{2}} + \mathrm{D}1^{\frac{1}{2}} . b^2 z^2 + \underset{c}{\mathrm{D}}^2 1^{\frac{1}{2}} b^4 z^4 + \&c. \right\}$

* I have, in a succeeding page, deduced this theorem of FAGNANI from the general method, contained in the following pages, for computing $\int dx \sqrt{\frac{1-e^2 x^2}{1-x^2}}$.

the $(n+1)$th term $= \underset{c}{\mathrm{D}}^n\, 1\tfrac{1}{2}\,.\, \frac{b^{2n}\, z^{2n}\, dz}{(1+z^2)^{\frac{3}{2}}} = \underset{c}{\mathrm{D}}^n\, 1\tfrac{1}{2}.\, b^{2n}.\, (2n-1)\, \frac{z^{2n-2}\, dz}{\sqrt{(1+z^2)}}$
$- \underset{c}{\mathrm{D}}^n\, 1\tfrac{1}{2}\, b^{2n} \times d\left(\frac{z^{2n-1}}{\sqrt{(1+z^2)}}\right)$. Now, if the integrals of
$\underset{c}{\mathrm{D}}^n\, 1\tfrac{1}{2}\, b^{2n}\, (2n-1)\, \frac{z^{2n-2}\, dz}{\sqrt{(1+z^2)}}$ be taken and added together, for the several values of n, (similarly to what has been already done,) there results,

$$f = {}^*\left\{\mathrm{D}1\tfrac{1}{2}.\, b^2 + \underset{c}{\mathrm{D}}^2\, 1\tfrac{1}{2}\,.\, \mathrm{D}1^{-\frac{3}{2}}.\, b^4 + \underset{c}{\mathrm{D}}^3\, 1\tfrac{1}{2}.\, \underset{c}{\mathrm{D}}^2\, 1^{-\frac{3}{2}}.\, b^6 + \&c.\right\}$$
$$\left\{ l.\, z + \sqrt{(1+z^2)} - \frac{z}{\sqrt{(1+z^2)}} \right\}$$
$$+ \underset{c}{\mathrm{D}}^2\, 1\tfrac{1}{2}\, b^4 .\, \frac{z}{\sqrt{(1+z^2)}} \left(\frac{z^2}{2}\right)$$
$$+ \underset{c}{\mathrm{D}}^3\, 1\tfrac{1}{2}\, b^6 .\, \frac{z}{\sqrt{(1+z^2)}} \left\{ \frac{z^4}{4} - \frac{5\,.\,z^2}{4\,.\,2} \right\}$$
$$+ \underset{c}{\mathrm{D}}^4\, 1\tfrac{1}{2}\, b^8 .\, \frac{z}{\sqrt{(1+z^2)}} \left\{ \frac{z^6}{6} - \frac{5z^4}{6\,.\,4} + \frac{5\,.\,3\,.\,z^2}{6\,.\,4\,.\,2} \right\}$$
$$+ \&c.$$

consequently, putting $z = \frac{1}{\sqrt{b}}$, we have $f\,(1) = 2f - 1 + b$

$$= 2\left(\mathrm{D}1\tfrac{1}{2}\, b^2 + \underset{c}{\mathrm{D}}^2\, 1\tfrac{1}{2}\,.\, \mathrm{D}1^{-\frac{3}{2}}\, b^4 + \underset{c}{\mathrm{D}}^3\, 1\tfrac{1}{2}\,.\, \underset{c}{\mathrm{D}}^2\, 1^{-\frac{3}{2}}.\, b^6 + \&c.\right\} \left\{ l.\, \frac{1+\sqrt{1+b}}{\sqrt{b}} - \frac{1}{\sqrt{1+b}} \right\}$$
$$+ \frac{2}{\sqrt{1+b}}$$
$$+ 2\underset{c}{\mathrm{D}}^2\, 1\tfrac{1}{2}\, \frac{1}{\sqrt{(1+b)}} \,.\, \frac{b^3}{2}$$
$$+ \frac{2\underset{c}{\mathrm{D}}^3\, 1\tfrac{1}{2}}{\sqrt{(1+b)}} \left\{ \frac{b^4}{4} - \frac{5b^3}{4\,.\,2} \right\}$$
$$+ \&c.$$

the series $\mathrm{D}1\tfrac{1}{2}\, b^2 + \underset{c}{\mathrm{D}}^2\, 1\tfrac{1}{2}\,.\, \mathrm{D}1^{-\frac{3}{2}}\, b^4 +$ &c. numerically expressed, is
$\frac{1}{2}\,.\, b^2 + \frac{1\,.\,1}{2\,.\,4}\,.\, \frac{3}{2}\,.\, b^4 + \frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\,.\, \frac{3\,.\,5}{2\,.\,4}\, b^6 +$ &c.

* This series is the same as
$$-\left\{\mathrm{D}1^{-\frac{1}{2}} b^2 + \mathrm{D}1^{-\frac{1}{2}}.\, \underset{c}{\mathrm{D}}^2\, 1^{-\frac{1}{2}}\, b^4 + \underset{c}{\mathrm{D}}^2\, 1^{-\frac{1}{2}}.\, \underset{c}{\mathrm{D}}^3\, 1^{-\frac{1}{2}}\, b^6 + \&c.\right\}$$

Let $y = l\,(1+\sqrt{(1+x)})$, then, $dy = \frac{dx}{2\sqrt{(1+x)}\,(1+\sqrt{(1+x)})}$

$$= \frac{dx}{2x} - \frac{dx}{2x\sqrt{(1+x)}}$$

$$= \frac{dx}{2x} - \frac{dx}{2x}\left\{1^{-\frac{1}{2}} - \frac{x}{2} + \frac{1\cdot 3}{2\cdot 4}x^2 - \text{\&c.}\right\}$$

$$\therefore y = \frac{x}{4} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^2}{4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{6} - \text{\&c.} + \text{corr.}$$

when $x = 0$ $y = l2 = \therefore$ corr.

hence, $l\,\frac{1+\sqrt{(1+b)}}{\sqrt{b}} - \frac{1}{\sqrt{(1+b)}}$

$$= \frac{b}{4} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{b^2}{4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{b^3}{6} + \text{\&c.}$$

$$-\left\{1 - \frac{b}{2} + \frac{1\cdot 3}{2\cdot 4}\cdot b^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}b^3 + \text{\&c.}\right\}$$

$$+ l2 + l\frac{1}{\sqrt{b}}$$

$$= l\,\frac{2}{\sqrt{b}} - 1 + \frac{3\cdot b}{4} - \frac{3\cdot 5}{2\cdot 4}\cdot\frac{b^2}{4} + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\cdot\frac{b^3}{6} - \text{\&c.}$$

If this series be substituted for $l\,\frac{1+\sqrt{(1+b)}}{\sqrt{b}} - \frac{1}{\sqrt{(1+b)}}$, in the above form for $f(1)$, if $\frac{1}{\sqrt{(1+b)}}$ be expanded, and the terms affected with like powers of b, be collected, we shall have the same series as LEGENDRE has given. EULER, however, is the original author of the series; and has expressed its law much more clearly than the French mathematician. In the EULERI *Opuscula*, Berlin, 1750, p. 165, the author says, that the elliptic quadrant

$$= 1 + Ab^2 + Bb^4 + C\,b^6 + \text{\&c.}$$

$$-\left\{\alpha b^2 + \beta b^4 + \gamma b^6 + \text{\&c.}\right\}\ \text{log. } b, \text{ in which}$$

$A = \log. 2 - \frac{1}{4}$	$\alpha = \frac{1}{2}$
$B = \frac{1\cdot 3}{2\cdot 4}A - \frac{1}{2}(\alpha - \beta) + \frac{1}{2}\cdot\frac{\beta}{2}$	$\beta = \frac{1\cdot 3}{2\cdot 4}\cdot\alpha$
$C = \frac{3\cdot 5}{4\cdot 6}B - \frac{1}{3}(\beta - \gamma) + \frac{1}{4}\cdot\frac{\gamma}{3}$	$\gamma = \frac{3\cdot 5}{4\cdot 6}\beta$
$D = \frac{5\cdot 7}{6\cdot 8}C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6}\cdot\frac{\delta}{4}$	$\delta = \frac{5\cdot 7}{6\cdot 8}\gamma$
$E = \frac{7\cdot 9}{8\cdot 10}D - \frac{1}{5}(\delta - \varepsilon) + \frac{1}{8}\cdot\frac{\varepsilon}{5}$	$\varepsilon = \frac{7\cdot 9}{8\cdot 10}\delta$
&c.	&c.

LEGENDRE's series is easily reducible to this, since $\log.\frac{2}{\sqrt{b}} = \frac{1}{2}\log.\frac{4}{b} = \log.2 - \frac{1}{2}\log.b$.

This memoir of EULER (*Animadversiones in Rectificationem Ellipseos*) is curious, on account of the strange artifices used to obtain the series for the length of the eccentric ellipse. It is characteristical of the peculiar mathematical powers of EULER, and also bears strong marks of the rapidity and eagerness with which he conducted every work of calculation. The author discovers the series and its law, partly by tentative methods, and partly by the use of a differential equation of the second order; and indeed, without the use of such an equation, it is difficult to exhibit the law. Let $f(1)$ represent the whole integral of f, from $x = 0$ to $x = 1$, then,

$$\frac{(1-b^2)\cdot d^2 f(1)}{db^2} - \frac{1+b^2}{b}\cdot\frac{df(1)}{db} + f(1) = 0.$$

Assume then, $f(1) = 1 + Ab^2 + Bb^4 + Cb^6 + \&c.$

$$+\left\{\alpha b^2 + \beta b^4 + \gamma b^6 + \&c.\right\}\log.b;$$

deduce the values of $\frac{d^2 f(1)}{db^2}$, $\frac{df(1)}{db}$; compare the terms affected with like powers of b; and the law of the series, such as it has been exhibited, may be deduced.

The following is the method of deducing the differential equation; $df = dz\sqrt{\frac{(1+b^2 z^2)}{(1+z^2)^{\frac{3}{2}}}}$ $\therefore \frac{df}{dz} = \frac{\sqrt{(1+b^2 z^2)}}{(1+z^2)^{\frac{3}{2}}}$; and, taking the partial differentials,

$$\frac{d^2 f}{dz.db} = \frac{bz^2}{\sqrt{(1+b^2 z^2)}\,(1+z^2)^{\frac{3}{2}}};$$

consequently, $\frac{df}{dz} = \frac{1}{\sqrt{(1+b^2 z^2)}\,(1+z^2)^{\frac{3}{2}}} + \frac{b\,d^2 f}{dz\,.\,db}$.

and $\frac{d^2 f}{dz\,.\,db} = \frac{-bz^2}{\sqrt{(1+b^2 z^2)^{\frac{3}{2}}}\,(1+z^2)^{\frac{3}{2}}} + \frac{d^2 f}{dz\,.\,db} + \frac{b\,.\,d^3 f}{dz\,.\,db^2}$

$$\therefore \frac{b^2 z^2}{(1+b^2 z^2)^{\frac{3}{2}}\,(1+z^2)^{\frac{3}{2}}} = \frac{b^2\,d^3 f}{dz\,.\,db^2};$$

or $\frac{1+b^2 z^2}{(1+b^2 z^2)^{\frac{3}{2}}(1+z^2)^{\frac{3}{2}}} - \frac{1}{(1+b^2 z^2)^{\frac{3}{2}}(1+z^2)^{\frac{3}{2}}} = \frac{b^2 d^3 f}{dz \,.\, db^2}$,

or $\frac{df}{dz} - \frac{bd^2 f}{dz \,.\, db} - \frac{1}{(1+b^2 z^2)^{\frac{3}{2}}(1+z^2)^{\frac{3}{2}}} = \frac{b^2 d^3 f}{dz \,.\, db^2}$.

Now, the differential of $\frac{z}{\sqrt{(1+z^2)(1+b^2 z^2)}} = dz \frac{(1-b^2 z^4)}{(1+z^2)^{\frac{3}{2}}(1+b^2 z^2)^{\frac{3}{2}}}$

$\therefore \frac{(b^2-1)\,dz}{b^2(1+z^2)^{\frac{3}{2}}(1+b^2 z^2)^{\frac{3}{2}}} = d\left\{\frac{z}{\sqrt{(1+z^2)(1+b^2 z^2)}}\right\} + \frac{df}{b^2} + \frac{2dz}{b^2 .\, \overline{1+z^2}^{\frac{3}{2}} \sqrt{1+b^2 z^2}}$

$\therefore \int \frac{b^2-1}{b^2} \cdot \frac{dz}{(1+z^2)^{\frac{3}{2}}(1+b^2 z^2)^{\frac{3}{2}}} = \frac{z}{\sqrt{(1+z^2)(1+b^2 z^2)}} + \frac{f}{b^2} - \frac{2f}{b^2} + \frac{2df}{b \,.\, db}$

$\therefore f - \frac{b \,.\, df}{db} - \frac{b^2}{b^2-1} \frac{z}{\sqrt{(1+z^2)(1+b^2 z^2)}} + \frac{f}{b^2-1} - \frac{2b}{b^2-1}\frac{df}{db} = \frac{b^2 d^2 f}{db^2}$

$\therefore f - \frac{1+b^2}{b} \cdot \frac{df}{db} + (1+b^2)\frac{d^2 f}{db^2} = 0$; since, when $x = 1$,

$z = \infty$, and $\frac{z}{\sqrt{(1+z^2)(1+b^2 z^2)}} = \frac{1}{bz} = 0$.

In order to compute the integral $\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ (f), when e is nearly $= 1$, by a series ascending by the powers of $\sqrt{(1-e^2)}$, it has been found necessary to establish this formula,

$$\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)} + \int dv \sqrt{\left(\frac{1-e^2 v^2}{1-v^2}\right)} = f(1) + \frac{e^2 x\sqrt{(1-x^2)}}{\sqrt{(1-e^2 x^2)}}.$$

Now, this formula, an analytical artifice useful for computation, applied to a particular curve, and translated into geometrical language, exhibits a curious property of the curve; thus, in an ellipse whose semiaxes are 1, $\sqrt{(1-e^2)}$, $\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, $\int dv \sqrt{\left(\frac{1-e^2 v^2}{1-v^2}\right)}$, represent arcs (E, E') corresponding to abscissas, x, v; and $f(1)$ is the elliptic quadrant $(E(1))$; hence

$$E + E' = E(1) + e^2 x \sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)}$$

$$\therefore E - \{E(1) - E'\} = e^2 x \sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)};$$

or the difference of two arcs, one reckoned from the extremity of the conjugate, the other from the extremity of the transverse, is equal to a right line, represented by $e^2 x \sqrt{\left(\frac{1 \quad x^2}{1-e^2 x^2}\right)}$.

This theorem is known by the name of FAGNANI's theorem.*

When $x = v$, or the quantity $e^2 x \sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)}$ is at its maximum,

$2f = f(1) + 1 - b$,

or $f - (f(1) - f) = 1 - b$,

or $E - \left\{E(1) - E'\right\} = 1 - b$;

or the elliptic quadrant is divided in such a manner, that the difference of the two arcs = difference of the semiaxes.

From the preceding analysis it is clear, that the computation of the integral of $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ is perfectly independent of the existence of the ellipse and its properties. But it also appears, that the property of the bisection of the ellipse, established geometrically, ought not to be regarded as a merely curious and beautiful property, since, by its aid, the length of the elliptic quadrant may be computed. Several other properties, considered hitherto in the light of curious and speculative truths, translated, would appear analytical artifices, and in computation practically useful.

By the preceding series, the integral of $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ may be computed, when e is nearly $= 0$ or 1. It is necessary, however, to possess a method of computing the integral when e is of mean value; and the methods I am about to exhibit, are such as to supersede the use of the two series ascending by the powers of e and b; in other words, from two similar methods, in all values of e between 0 and 1, the integral may be commodiously computed. The principle of the method is this, if $df = dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$; then,

* This theorem of FAGNANI has lately been very neatly demonstrated, by a most skilful mathematician, Mr. BRINKLEY, in the Irish Transactions, by a geometrical process, but not without the use of prime and ultimate ratios. Indeed, the nature of the subject is such, that the theorem cannot be established, without the use of the fluxionary calculus, or of some calculus equivalent to it.

df', df'', df''', &c. being similar differential expressions, df may be resolved into $md\mathrm{P}+\alpha\,.\,df'+\beta\,.\,df''$, ($d\mathrm{P}$ being a perfect differential, m, α, β, &c. constant coefficients,) in like manner,

df' may be resolved into $m'd\mathrm{P}'+\alpha'\,.\,df''+\beta'\,.\,df'''$,
df'' into - - $m''d\mathrm{P}''+\alpha''\,.\,df'''+\beta''\,.\,df^{\text{IV}}$,
&c.

and, consequently, df may be resolved into

$$m.d\mathrm{P}+\alpha m'\,.\,d\mathrm{P}'+\alpha\,\alpha'\,m'\,m''\,d\mathrm{P}''+\text{\&c.}$$
$$+\alpha\alpha'\alpha''\alpha'''\ \text{\&c.}\ df^{''\,\text{\&c.}}+\text{\&c.}$$

This resolution depends on a very simple, and, if I may use the term, natural substitution, in the form $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, of which, to the best of my knowledge, M. LAGRANGE is the author.

Let $y=x\sqrt{\left(\frac{1-x^2}{1-e^2x^2}\right)}$;

then, $x^2=\frac{e^2y^2+1}{2}-\frac{1}{2}\sqrt{(1+2\,(e^2-2)\,y^2+e^4y^4)}$,

and $\frac{dx}{\sqrt{(1-x^2)\,(1-e^2x^2)}}=\frac{dy}{1-2x^2+e^2y^2}=\frac{dy}{\sqrt{(1+2\,.\,(e^2-2)\,y^2+e^4y^4)}}$.

Now, if $p=1+\sqrt{(1-e^2)}$, $p^2=2-e^2+2\sqrt{(1-e^2)}$,
if $q=1-\sqrt{(1-e^2)}$, $q^2=2-e^2-2\sqrt{(1-e^2)}$,

and $1+2\,.\,(e^2-2)\,y^2+e^4y^4=(1-p^2y^2)\,.\,(1-q^2y^2)$

$$\therefore\ \frac{dx}{\sqrt{(1-x^2)\,(1-e^2x^2)}}=\frac{dy}{\sqrt{(1-p^2y^2)\,(1-q^2y^2)}}=\frac{du'}{p\,.\,\sqrt{(1-u'^2)\,(1-\frac{q^2}{p^2}u'^2)}};$$

putting $y=\frac{u'}{p}$, and putting $\frac{q}{p}=e'$,

$\frac{dx}{\sqrt{(1-x^2)\,(1-e^2x^2)}}$ is transformed into $\frac{du'}{p\sqrt{(1-u'^2)\,(1-e'^2u'^2)}}$;

similarly, putting $p'=1+\sqrt{(1-e'^2)}$, $e''=\frac{q'}{p'}=\frac{1-\sqrt{(1-e'^2)}}{1+\sqrt{(1-e'^2)}}$,

and $u''=p'\,u'\sqrt{\left(\frac{1-u'^2}{1-e'^2u'^2}\right)}$,

$$\frac{du'}{\sqrt{(1-u'^2)\,(1-e'^2u'^2)}}=\frac{1}{p'}\,.\,\frac{du''}{\sqrt{(1-u''^2)\,(1-e''^2u''^2)}}.$$

Hence, since $e'=\frac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}}$, $p=1+\sqrt{(1-e^2)}=\frac{2}{1+e'}$,

$\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$ may be transformed into

$\frac{1+e'}{2} \cdot \frac{du'}{\sqrt{(1-u'^2)(1-e'^2u'^2)}}$, or $\frac{(1+e')(1+e'')}{2 \quad . \quad 2} \cdot \frac{du''}{\sqrt{(1-u''^2)(1-e''^2u''^2)}}$;

or into $\frac{(1+e')(1+e'')(1+e''') \quad \ldots\ldots \quad 1+e^{(n)}}{2 \quad . \quad 2 \quad . \quad 2 \quad \ldots\ldots \quad 2} \cdot \frac{du^{(n)}}{\sqrt{(1-u^{(n)2})(1-e^{(n)2}u^{(n)2})}}$.

And, similarly, putting $U' = \sqrt{(1-u'^2)(1-e'^2u'^2)}$,

$U'' = \sqrt{(1-u''^2)(1-e''^2u''^2)}$ &c.

$\frac{(A+Bx^2) \cdot dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, may be transformed into

$\left(\frac{2A+B}{2p}\right) \cdot \frac{du'}{U'} - \frac{B}{2p} \cdot du' + \frac{Be^2}{2p^3} \cdot \frac{u'^2 . du'}{U'}$;

or, to render the last term like the original form, into

$\left(\frac{2A+B}{2p} - \frac{Be^2 A'}{2p^3 B'}\right) \frac{du'}{U'} - \frac{B}{2p} du' + \frac{Be^2}{2p^3 B'} (A'+B'u'^2) \frac{du'}{U'}$.

And, into a form exactly similar may $(A'+B'u'^2) \cdot \frac{du'}{\sqrt{(1-u'^2)(1-e'^2u'^2)}}$ be transformed.

Hence, to transform df or $dx \sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)} = \frac{dx (1-e^2x^2)}{\sqrt{(1-x^2)(1-e^2x^2)}}$

$A = 1$, $B = -e^2$; consequently,

$df = \frac{e^2}{2p} \cdot du' - \frac{\sqrt{(1-e^2)}}{p} \cdot \frac{du'}{U'} + \frac{p}{2} (1-e'^2u'^2) \cdot \frac{du'}{U'}$;

or, since $\sqrt{(1-e^2)} = \frac{1-e'}{1+e'}$ and $\frac{p}{2} = \frac{1}{1+e'}$

$$df = \frac{e^2}{4}(1+e')\,du' - \frac{1-e'}{2} \cdot \frac{du'}{U'} + \frac{1}{1+e'} \cdot df'; \qquad (a)$$

similarly, $df' = \frac{e'^2}{4} \cdot (1+e'')\,du'' - \left(\frac{1-e''}{2}\right) \frac{du''}{U''} + \frac{1}{1+e''} \cdot df''$;

$df'' = \frac{e''^2}{4}(1+e''')\,du''' -$ &c.

The utility of this transformation will appear, by observing that the quantities e', e'', e''', &c. continually decrease; thus,

$e' = \frac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}} = \frac{e^2}{(1+\sqrt{(1-e^2)})^2} = e \cdot \frac{e}{(1+\sqrt{(1-e^2)})^2}$.

Hence, if e be a fraction, $e' = e \times$ a fraction; consequently, e' is $\angle\, e$; similarly, e'' is $\angle\, e'$, $e''' \angle\, e''$ &c; hence, if the series for $\int dx \sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ does not converge quickly, transform df as above,

and the series for $\frac{du'}{U'}$, df', converge more quickly; but, if not with sufficient rapidity, again transform $\frac{du'}{U'}$, df', and the resulting forms $\frac{du''}{U''}$, df'', may be converted into series of still greater convergency; so that, by this method, we may proceed with certainty to the computation of $\int dx \sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$. If we stop at the first transformation, there results a series for f, the same as is given in a very able memoir of Mr. IVORY's, inserted in the Edinburgh Transactions, Vol. IV. p. 178.

Thus, $df = \frac{e^2}{4}(1+e')\,du' - \frac{(1-e')}{2} \cdot \frac{du'}{U'} + \frac{1}{1+e'} \cdot df'$, $\qquad (a)$

$$\text{or} = \frac{e^2}{4} \cdot (1+e')\,du' + \frac{1+e'^2}{2 \cdot (1+e')}(A)\frac{du'}{U'} - \frac{e'^2}{1+e'}(B) \cdot \frac{u'^2\,du'}{U'}.$$

Now, $u' = px\sqrt{\left(\frac{1-x^2}{1-e^2x^2}\right)}$, which quantity is at its maximum when $x = \frac{1}{\sqrt{(1+b)}}$, and then $u'=1$; consequently, whilst x from 0 becomes 1, u' from 0 passes through its maximum (1) to 0 again; consequently, $\int \frac{du'}{U'}$ from $x=0$ to $x=1$, $= 2\int \frac{du'}{U'}$ from $u'=0$ to $u'=1$.

Now, $\int A \cdot \frac{du'}{U'} - \int B \cdot \frac{u'^2\,du'}{U'}$

$$= \int \frac{A\,du'}{\sqrt{(1-u'^2)}}\left\{1^{-\frac{1}{2}} - D1^{-\frac{1}{2}} \cdot e'^2u'^2 + \underset{c}{D}^2\,1^{-\frac{1}{2}} \cdot e'^4u'^4 - \&c.\right\}$$

$$- \int \frac{Bu'^2 \cdot du'}{\sqrt{(1-u'^2)}}\left\{1^{-\frac{1}{2}} - D1^{-\frac{1}{2}}\,e'^2u'^2 + \underset{c}{D}^2\,1^{-\frac{1}{2}} \cdot e'^4u'^4 - \&c.\right\}$$

But, by a preceding form, page 227,

$$\int \frac{u'^{2n} \cdot du'}{\sqrt{(1-u'^2)}} = -\sqrt{(1-u'^2)}\left\{\frac{u'^{2n-1}}{2n} + \frac{2n-1 \cdot u'^{2n-3}}{2n\,(2n-2)} - \&c.\right\} + \frac{(2n-1)\,(2n-3)\,\ldots\,5 \cdot 3}{2n \cdot (2n-2)\,\ldots\,4 \cdot 2}\int \frac{du'}{\sqrt{1-u'^2}};$$

put $u'=1$, and all the terms vanish, except the last; consequently, from $u'=0$ to $u'=1$,

$$\int \pm \underset{c}{D}^n\,1^{-\frac{1}{2}}\,e'^{2n} \cdot \frac{u'^{2n}\,du'}{\sqrt{(1-u'^2)}} = \pm \underset{c}{D}^n\,1^{-\frac{1}{2}} \cdot e'^{2n} \times \frac{(2n-1)\,(2n-3)\,\ldots\,5 \cdot 3 \cdot 1}{2n \cdot (2n-2)\,\ldots\,6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$\left(\frac{\pi}{2} = \frac{3 \cdot 14159+}{2}\right) = \pm \underset{c}{D}^n 1^{-\frac{1}{2}} \times \pm \underset{c}{D}^n 1^{-\frac{1}{2}} \cdot e'^{2n} \cdot \frac{\pi}{2} = \therefore$, whether n be even or odd, $\left(\underset{c}{D}^n 1^{-\frac{1}{2}}\right)^2 e'^{2n} \cdot \frac{\pi}{2}$;

similarly, $\int \pm \underset{c}{D}^n 1^{-\frac{1}{2}} \cdot e'^{2n} \cdot u'^{2n+2} \cdot \frac{du'}{\sqrt{(1-u'^2)}} = -\underset{c}{D}^n 1^{-\frac{1}{2}} \cdot \underset{c}{D}^{n+1} 1^{-\frac{1}{2}} e'^{2n} \cdot \frac{\pi}{2}$

Hence, putting for n the several values 0, 1, 2, 3, 4, &c. the sum of the integrals from $u' = 0$ to $u' = 1$

$$= \frac{A\pi}{2}\left\{1^{-\frac{1}{2}} + \left(D 1^{-\frac{1}{2}}\right)^2 e'^2 + \left(\underset{c}{D}^2 1^{-\frac{1}{2}}\right)^2 e'^4 + \&c.\right\}$$

$$+ \frac{B\pi}{2}\left\{D 1^{-\frac{1}{2}} + D 1^{-\frac{1}{2}} \cdot \underset{c}{D}^2 1^{-\frac{1}{2}} e'^2 + \underset{c}{D}^2 1^{-\frac{1}{2}} \cdot \underset{c}{D}^3 1^{-\frac{1}{2}} e'^4 + \&c.\right\}$$

or, putting for A and B, their values $\frac{1+e'^2}{2 \cdot (1+e')}$, $\frac{2e'^2}{2 \cdot (1+e')}$, the integral

$$= \frac{\pi}{2 \cdot 2 \cdot (1+e')}\left\{1 + \begin{array}{c} 1^{-\frac{1}{2}} \\ 2D1^{-\frac{1}{2}} \\ +\left(D1^{-\frac{1}{2}}\right)^2 \end{array}\middle| e'^2 \quad \begin{array}{c} \left(D1^{-\frac{1}{2}}\right)^2 \\ +2D1^{-\frac{1}{2}} \cdot \underset{c}{D}^2 1^{-\frac{1}{2}} \\ \left(\underset{c}{D}^2 1^{-\frac{1}{2}}\right)^2 \end{array}\middle| e'^4 + \&c.\right.$$

and, generally, the coefficient affected with e'^{2n} is

$$\left(\underset{c}{D}^{n-1} 1^{-\frac{1}{2}}\right)^2 + 2\underset{c}{D}^{n-1} 1^{-\frac{1}{2}} \cdot \underset{c}{D}^n 1^{-\frac{1}{2}} + \left(\underset{c}{D}^n 1^{-\frac{1}{2}}\right)^2 = \left\{\underset{c}{D}^{n-1} 1^{-\frac{1}{2}} + \underset{c}{D}^n 1^{-\frac{1}{2}}\right\}^2,$$

but $-(2n-1)\,\underset{c}{D}^n 1^{\frac{1}{2}} = \underset{c}{D}^n 1^{-\frac{1}{2}}$,

and $2n \cdot \underset{c}{D}^n 1^{\frac{1}{2}} = \underset{c}{D}^{n-1} 1^{-\frac{1}{2}}$

$$\therefore \underset{c}{D}^n 1 = \underset{c}{D}^{n-1} 1^{-\frac{1}{2}} + \underset{c}{D}^n 1^{-\frac{1}{2}}.$$

Hence, the coefficient affected with e'^{2n} is $\left(\underset{c}{D}^n 1^{\frac{1}{2}}\right)^2$; and, consequently, the integral from $u' = 0$ to $u' = 1$

$$= \frac{\pi}{2 \cdot 2 \cdot 1+e'}\left\{1 + \left(D1^{\frac{1}{2}}\right)^2 \cdot e'^2 + \left(\underset{c}{D}^2 1^{\frac{1}{2}}\right)^2 \cdot e'^4 + \left(\underset{c}{D}^3 1^{\frac{1}{2}}\right)^2 \cdot e'^6 + \&c.\right\}$$

the double of this is the integral (f) from $x = 0$ to $x = 1$,

or $\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ from $x = 0$ to $x = 1$

$$= \frac{\pi}{2(1+e')}\left\{1 + \left(D1^{\frac{1}{2}}\right)^2 \cdot e'^2 + \left(\underset{c}{D}^2 1^{\frac{1}{2}}\right)^2 \cdot e'^4 + \left(\underset{c}{D}^3 1^{\frac{1}{2}}\right)^2 \cdot e'^6 + \&c.\right\}$$

or, developing the symbols $D1^{\frac{1}{2}}$ &c.

the integral

$$= \frac{\pi}{2 \cdot (1+e')} \left\{ 1 + \frac{1^2}{2^2} \cdot e'^2 + \frac{1^2 \cdot 1^2}{2^2 \cdot 4^2} \cdot e'^4 + \frac{1^2 \cdot 1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} e'^6 + \&c. \right\}. \quad (4)$$

Which is the same series as is given in the Edinburgh Transactions, Vol. IV. p. 178, and which its ingenious author, Mr. IVORY, derived from a method of LAGRANGE, contained in the Berlin Acts for 1784. According to that method, $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ is put under the form $A d\theta \left\{ 1 + a^2 + 2a \cos. 2\theta \right\}^{\frac{1}{2}}$, and its exponential expression substituted for cos. 2θ.

I have deduced the preceding series ascending by the powers of e' or of $\frac{1-b}{1+b}$, in order to show, that it is a particular result of the general method of the transformation of $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$. For purposes of computation, it will be convenient to push the transformation farther; if, for instance, to quantities involving e'', the integral of $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ from $x = 0$ to $x = 1$, may be computed from 2 series; or the whole integral equals

$$\frac{\pi}{(1+e')(1+e'')} \left\{ 1^{\frac{1}{2}} + (\mathrm{D} 1^{\frac{1}{2}})^2 \cdot e''^2 + \left(\frac{\mathrm{D}^2}{c} 1^{\frac{1}{2}}\right)^2 e''^4 + \&c. \right\}$$

$$- \frac{2\pi \cdot (1-e')(1+e'')}{4} \left\{ 1^{-\frac{1}{2}} + (\mathrm{D} 1^{-\frac{1}{2}})^2 e''^2 + (\mathrm{D}'' 1^{-\frac{1}{2}})^2 e''^4 + \&c. \right\}$$

which expression may be derived after a manner precisely similar to that by which I have deduced the series ascending by the powers of e'.

If the transformation of df be indefinitely continued, there results a form very convenient for the computation of the integral of $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, in all values of e between 0 and $\sqrt{\frac{1}{2}}$; thus,

$$df = \frac{e'}{1+e'} \cdot du' - \frac{1-e'}{2} \cdot \frac{du'}{\mathrm{U}'} + \frac{df'}{1+e'}, \quad (a)$$

$$\text{or} = \frac{e'}{1+e'} \cdot du' + \left\{ \frac{1+e'}{2} - 1 \right\} \frac{du'}{\mathrm{U}'} + \frac{df'}{1+e'};$$

similarly, $df' = \frac{e''}{1+e''} \cdot du'' + \left\{ \frac{1+e''}{2} - 1 \right\} \cdot \frac{du''}{\mathrm{U}''} + \frac{df''}{1+e''}$.

Hence,

$$df = \frac{e'}{1+e'} \cdot du' + \frac{e''}{(1+e')(1+e'')} \cdot du''$$

$$+ \frac{1+e'}{2} \cdot \frac{du'}{U'} + \frac{1+e''}{2} \cdot \frac{1}{1+e'} \cdot \frac{du''}{U''}$$

$$- \left\{ \frac{du'}{U'} + \frac{1}{1+e'} \cdot \frac{du''}{U''} \right\} + \frac{1}{(1+e')(1+e'')} df''$$

$$= \frac{e'}{1+e'} \cdot du' + \frac{e''}{(1+e')(1+e'')} \cdot du'' + \frac{e'''}{(1+e')(1+e'')(1+e''')} du''' + \text{\&c.}$$

$$+ \frac{1+e'}{2} \cdot \frac{du'}{U'} + \frac{1+e''}{2} \cdot \frac{1}{1+e'} \cdot \frac{du''}{U''} + \frac{1+e'''}{2 \quad (1+e') \ 1+e''} \cdot \frac{du'''}{U'''} + \text{\&c.}$$

$$- \left\{ \frac{du'}{U'} + \frac{1}{1+e'} \cdot \frac{du''}{U''} + \frac{1}{(1+e')(1+e'')} \cdot \frac{du'''}{U'''} + \text{\&c.} \right\}$$

$$+ \frac{1}{(1+e')(1+e'') \ldots \text{\&c.}}, df^{(n)}.$$

Now, $\frac{du'}{U'} = \frac{1+e''}{2} \cdot \frac{du''}{U''} = \frac{(1+e'')(1+e''')}{2 \ . \ 2} \cdot \frac{du'''}{U'''}$

$= \frac{(1+e'')(1+e''') \ldots\ldots (1+\varepsilon)}{2^n} \cdot \frac{dv}{V}$, ($\varepsilon$, v, V, representing the last terms of series, e', e'', e''', &c; u', u'', u''', &c; U$'$, U$''$, U$'''$, &c.),

$$\frac{\varepsilon}{(1+e')(1+e'') \ldots 1+\varepsilon} = \frac{e^2 . e' \, e'' . \text{\&c.} \ (1+e')(1+e'') \ldots (1+\varepsilon)}{4 \cdot 4 \cdot 4 \cdot \text{\&c.}},$$

and, $df^{(n)} = \frac{dv}{V} - \frac{\varepsilon^2 v^2 dv}{V}$; let $P = (1+e')(1+e'') \ldots\ldots (1+\varepsilon)$; then,

$$df = \frac{e^2}{4} . (1+e') . du' + \frac{e^2 . e'(1+e') . (1+e'')}{4 \cdot 4} . du'' + \frac{e^2 . e' . e'' . (1+e')(1+e'')(1+e''')}{4 \cdot 4 \cdot 4}$$

$$+ \text{\&c.} \ (d \Sigma u')$$

$$+ \frac{P}{2^n} \left\{ 1 + \frac{2}{(1+e')^2} + \frac{2^2}{(1+e')^2 (1+e'')^2} + \frac{2^n}{P^2} + \text{\&c.} \right\} . \frac{dv}{V}$$

$$- \frac{P}{2^n} \left\{ 0 + \frac{2 \ .}{1+e'} + \frac{2^2}{(1+e')^2 (1+e'')} + \frac{2^n (1+e)}{P^2} + \text{\&c.} \right\} + \frac{\varepsilon^2 dv}{2^n P \ . \ V};$$

consequently,

$$df = d \Sigma u' + \frac{P}{2^n} . \frac{dv}{V}$$

$$- \frac{P}{2^n} \left\{ \frac{2e'}{(1+e')^2} + \frac{2^2 e''}{(1+e')^2 (1+e'')^2} + \text{\&c.} \right\} . \frac{dv}{V}$$

$$+ \frac{\varepsilon^2}{P} \cdot \frac{dv}{V};$$

and, consequently, since $\frac{2e'}{(1+e')^2} = \frac{e^2}{2}$, $\frac{2^2 e''}{(1+e')^2 (1+e'')^2} = \frac{e^2 \ e'}{2 \ . \ 2}$ &c.

(for e' being $= \frac{1-\sqrt{(1-e^2)}}{1-\sqrt{(1-e^2)}}$, $e^2 = \frac{4e'}{(1+e')^2}$),

$$f = \Sigma u' + \frac{P}{2^n} \int \frac{dv}{V}$$
$$- \frac{P}{2^n} \cdot \left\{ \frac{e^2}{2} + \frac{e^2 \cdot e'}{2 \cdot 2} + \frac{e^2 \cdot e' \cdot e''}{2 \cdot 2 \cdot 2} + \text{\&c.} \right\} \int \frac{dv}{V}$$
$$+ \frac{\varepsilon^2}{P} \int \frac{dv}{V}.$$

Now, $\frac{\varepsilon^2}{P} = \frac{e^2 \cdot e' \cdot e'' \cdot \text{\&c.} \ldots \varepsilon}{4 \cdot 4 \cdot 4 \quad \ldots 4} \cdot (1+e')(1+e'')(1+e''') \ldots (1+\varepsilon)$; consequently, since e', e'', e''', &c. continually decrease, the quantity $\frac{\varepsilon^2}{P} \int \frac{dv}{V}$ may be rejected, and $\frac{dv}{V} = \frac{dv}{\sqrt{(1-V^2)(1-e^2 V^2)}}$, nearly $= \frac{dv}{\sqrt{(1-V^2)}}$; consequently,

$$f = \frac{e^2}{4}(1+e')u' + \frac{e^2 \cdot e' \cdot (1+e')(1+e'')}{4 \cdot 4} u'' + \text{\&c.} \tag{5}$$
$$+ \frac{P}{2^n} \int \frac{dv}{\sqrt{(1-V^2)}} - \frac{P}{2^n} \left\{ \frac{e^2}{2} + \frac{e^2 \cdot e'}{2 \cdot 2} + \frac{e^2 \cdot e' \cdot e''}{2 \cdot 2 \cdot 2} + \text{\&c.} \right\} \int \frac{dv}{\sqrt{(1-V^2)}}.$$

When x passes from 0 to 1, u' passes from 0 to 1, (its maximum,) and from 1 to 0; similarly, when u' passes from 0 to 1, u'' passes from 0 to 1, (its maximum,) and from 1 to 0. Hence, $\int \frac{dx}{\sqrt{(1-x^2)}}$ generated from $x = 0$ to $x = 1 = 2\int \frac{du'}{\sqrt{(1-u'^2)}}$, from $u' = 0$ to $u' = 1$; $= 4\int \frac{du''}{\sqrt{(1-u''^2)}}$, from $u'' = 0$ to $u'' = 1$; $= 2^n \int \frac{dv}{\sqrt{(1-V^2)}}$, from $v = 0$ to $v = 1$; consequently, since u', u'', u''', &c. $= 0$, when $x = 1$, the whole integral of $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ from $x = 0$ to $x = 1$

$$= \frac{P\, 2^n}{2^n} \cdot \frac{\pi}{2} - \frac{P}{2^n} \left\{ \frac{e^2}{2} + \frac{e^2 \cdot e'}{2 \cdot 2} + \frac{e^2 \cdot e' \cdot e''}{2 \cdot 2 \cdot 2} + \text{\&c.} \right\} 2^n \cdot \frac{\pi}{2}$$
$$= \left(\text{putting } Q = \frac{e}{2} + \frac{e \cdot e'}{2 \cdot 2} + \frac{e \cdot e' \cdot e''}{2 \cdot 2 \cdot 2} + \text{\&c.}\right) P \cdot \frac{\pi}{2} - P e Q \frac{\pi}{2}$$
$$= P(1 - eQ)\frac{\pi}{2}. \tag{6}$$

Which is the same form as was first given by Mr. WALLACE, in the Edinburgh Transactions, Vol. V. p. 280.

The form (5) may easily be made to agree with that given, by the last mentioned author, for the length of an elliptic arc.

Thus, $u' = \frac{2}{1+e'} . x\sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)}$, $u'' = \frac{2}{1+e''}$, $u'\sqrt{\left(\frac{1-u'^2}{1-e'^2 u'^2}\right)}$ &c.

If we call, then, x the sine of an arc θ,

$$u' = \frac{2}{1+e'} \cdot \frac{\sin. \theta . \cos. \theta}{\sqrt{(1-e^2. \sin. \theta^2)}} = \frac{\sin. 2\theta}{(1+e')\sqrt{(1-e^2(\frac{1}{2}-\frac{1}{2}\cos. 2\theta))}}$$

$= \left(\text{since } \frac{e^2}{2} = \frac{2e'}{(1+e')^2}\right) \frac{\sin. 2\theta}{\sqrt{(1+e'^2+2e'. \cos. 2\theta)}}$; similarly, calling u' the sine of 2θ, u'' will equal $\frac{\sin. 4\theta'}{\sqrt{(1+e''^2+2e''. \cos. 4\theta')}}$, and so on; consequently, expressed in geometrical language,

$f = \frac{e^2}{4}(1+e')\sin. 2\theta' + \frac{e^2. e'(1+e')(1+e'')}{4 \cdot 4}. \sin. 4\theta'' + \text{\&c.}$

$+ P . \phi - PeQ\phi$, (where ϕ is the limit to which the arcs in the series θ', θ'', θ''', &c. approach,) and, consequently, since $\nu = 2^n\phi$, $\frac{d\nu}{\sqrt{(1-V^2)}} = 2^n . d\phi$, and $\frac{P}{2^n} . \int\frac{d\nu}{\sqrt{(1-V^2)}} = P\phi$.

Mr. WALLACE obtained his formula, following a method given by Mr. IVORY, in the fourth Volume of the Edinburgh Transactions; and both these ingenious authors have employed, probably without adopting, the substitution of LAGRANGE, and the principle of his transformation, such as that great mathematician uses in finding the integral of $\frac{P . dx}{\sqrt{(e+fx^2)(g+bx^2)}}$.

Since $e' = \frac{e^2}{(1+\sqrt{(1-e^2)})^2}$, $e'' = \frac{e'^2}{(1+\sqrt{(1-e'^2)})^2}$ &c.

When e is a small fraction, the quantities e', e'', e''', &c. decrease very rapidly; and, consequently, the preceding form is very commodious for the computation of $\int dx\sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, when e is any fraction between o and $\sqrt{\frac{1}{2}}$. It ceases, however, to be commodious when e is nearly $= 1$, or is not equally commodious with the series $1 + Ab^2 + Bb^4 + \text{\&c.}$

$+ \left\{\alpha b^2 + \beta b^4 + \text{\&c.}\right\}\log. b$, given page 234. I purpose, therefore, now to exhibit a form by which the integral of $dx\sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ may be conveniently computed, when e is any fraction between $\sqrt{\frac{1}{2}}$ and 1,

$df = dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$. Let $x = \frac{v}{\sqrt{(1+v^2)}}$, then, $df = \frac{dv\sqrt{(1+b^2v^2)}}{(1+v^2)^{\frac{3}{2}}}$

$= d\left(\frac{v}{\sqrt{(1+v^2)}}\sqrt{(1+b^2v^2)}\right) - \frac{b^2v^2\,dv}{\sqrt{(1+v^2)\,(1+b^2v^2)}}$.

Let $z = v\sqrt{\left(\frac{1+v^2}{1+b^2v^2}\right)}$, then $v^2 = \frac{b^2z^2 - 1 + \sqrt{(1+p^2z^2)\,(1+q^2z^2)}}{2}$;

since $(1+p^2z^2)\,(1+q^2z^2) = 1 + 2\,.\,(2-b^2)\,z^2 + b^4z^4$, when $p = 1+\sqrt{(1-b^2)}$ and $q = 1-\sqrt{(1-b^2)}$.

Hence, since $\frac{dv}{\sqrt{(1+v^2)\,(1+b^2v^2)}} = \frac{dz}{\sqrt{(1+p^2z^2)\,(1+q^2z^2)}}$

$= \frac{dz'}{p\sqrt{(1+z'^2)\,(1+{}^{\backprime}b^2z'^2)}\,(Z')}$, (putting $z = \frac{z'}{p}$, ${}^{\backprime}b = \frac{q}{p} = \frac{1-\sqrt{(1-b^2)}}{1+\sqrt{(1-b^2)}}$),

we have

$$\frac{b^2v^2\,dv}{\sqrt{(1+v^2)\,(1+b^2v^2)}} = \frac{b^2\,dz'}{p\,.\,Z'}\,.\left\{\frac{b^2z'^2}{2p^2} - \frac{1}{2} + \frac{1}{2}Z'\right\}$$

$$= \frac{b^2}{4}\,.\,(1+{}^{\backprime}b)\,dz' - \frac{b^2}{4}(1+{}^{\backprime}b)\frac{dz'}{Z'} + \frac{{}^{\backprime}b^2}{1+{}^{\backprime}b}\,.\,\frac{z'^2\,dz'}{Z'}.$$

Similarly, putting $z'' = {}^{\backprime}p\frac{z'\sqrt{1+z'^2)}}{\sqrt{1+{}^{\backprime}b^2z'^2)}}$, ${}^{\backprime\backprime}b = \frac{1-\sqrt{(1-{}^{\backprime}b^2)}}{1+\sqrt{(1-{}^{\backprime}b^2)}}$,

$\sqrt{(1+z''^2)\,(1+{}^{\backprime\backprime}b^2z''^2)} = Z''$,

$$\frac{{}^{\backprime}b^2z'^2.\,dz'}{Z'} = \frac{{}^{\backprime}b^2}{4}(1+{}^{\backprime\backprime}b)\,dz'' - \frac{{}^{\backprime}b^2}{4}(1+{}^{\backprime\backprime}b)\frac{dz''}{Z''} + \frac{{}^{\backprime\backprime}b^2}{1+{}^{\backprime\backprime}b}\,.\,\frac{z''^2\,dz''}{Z''},$$

&c. &c.

Consequently, since $\frac{dz'}{Z'} = \frac{1+{}^{\backprime\backprime}b}{2}\,.\,\frac{dz''}{Z''} = \frac{1+{}^{\backprime\backprime\backprime}b\,.\,1+{}^{\backprime\backprime}b}{2\quad.\quad2}\,.\,\frac{dz'''}{Z'''}$ &c.

$$df = d\left(v\sqrt{\left(\frac{1+b^2v^2}{1+v^2}\right)}\right)$$

$$-\left\{\frac{b^2}{4}(1+{}^{\backprime}b)\,dz' + \frac{{}^{\backprime}b^2}{4}\,.\,\frac{1+{}^{\backprime\backprime}b}{1+{}^{\backprime}b}\,.\,dz'' + \frac{{}^{\backprime\backprime}b^2}{4}\,\frac{1+{}^{\backprime\backprime\backprime}b}{(1+{}^{\backprime}b)\,(1+{}^{\backprime\backprime}b)}\,.dz''' + \text{\&c.}\right\}$$

$$+\left\{\begin{array}{l} +\frac{b^2}{4}\,.\,(1+{}^{\backprime}b).\frac{1+{}^{\backprime\backprime}b}{2}\,.\,\frac{1+{}^{\backprime\backprime\backprime}b}{2}\,.\,.\,.\,.\,.\,\frac{1+\beta}{2} \\ +\frac{{}^{\backprime}b^2}{4\,(1+{}^{\backprime}b)}\,.\,(1+{}^{\backprime\backprime}b)\,.\,\frac{1+{}^{\backprime\backprime\backprime}b}{2}\,.\,.\,.\,.\,.\,\frac{1+\beta}{2} \\ +\frac{{}^{\backprime\backprime}b^2}{4\,.\,(1+{}^{\backprime}b)\,(1+{}^{\backprime\backprime}b)}\,(1+{}^{\backprime\backprime\backprime}b)\,.\,.\,.\,.\,.\,\frac{1+\beta}{2} \\ +\text{\&c.} \end{array}\right\}\frac{d\zeta}{\sqrt{(1+\zeta^2)\,(1+\beta^2\zeta^2)}}$$

$$\frac{-\beta^2}{(1+{}^{\backprime}b)\,(1+{}^{\backprime\backprime}b)\,.....\,(1+\beta)},\ \frac{\zeta^2\,d\zeta}{\sqrt{(1+\zeta^2)\,(1+\beta^2\zeta^2)}};$$

β, ζ, being the last terms of the series ${}^{\backprime}b$, ${}^{\backprime\backprime}b$, ${}^{\backprime\backprime\backprime}b$, &c. z', z'', z''', &c. continued to n terms; put the product $(1+{}^{\backprime}b)\,(1+{}^{\backprime\backprime}b)\,....(1+\beta) = {}^{\backprime}\mathrm{P}$,

and $\frac{b}{2} + \frac{b\,.\,{}'b}{2\,.\,2} + \frac{b\,.\,{}'b\,.\,{}''b}{2\,.\,2\,.\,2} + \&c. = {}'Q$; then, since $\frac{\zeta^2\, d\zeta}{\sqrt{(1+\zeta^2)\,(1+\beta^2\,\zeta^2)}}$ nearly $= \frac{\zeta^2\, d\zeta}{\sqrt{(1+\zeta^2)}} = \frac{1}{2}\, d\,(\zeta\sqrt{(1+\zeta^2)}) - \frac{1}{2}\,\frac{d\zeta}{\sqrt{(1+\zeta^2)}}$,

and $\frac{\beta^2}{2\,.\,{}'P} = \frac{b^2\,.\,{}'b\,.\,{}''b\,\ldots\ldots\,\beta}{2\,.\,2\,.\,2\;\;\;\;\;.}\,.\,\frac{(1+{}'b)}{2}\,.\,\frac{(1+{}''b)}{2}\,.\ldots.\,\frac{(1+\beta)}{2}$,

we have

$$f = v\sqrt{\left(\frac{1+b^2\,v^2}{1+v^2}\right)} \tag{7}$$

$$-\left\{\frac{b^2}{2}\,.\,\frac{1+{}'b}{2}\,z' + \frac{b^2\,.\,{}'b}{2\,.\,2}\,.\,\frac{(1+{}'b)}{2}\,.\,\frac{(1+{}''b)}{2}\,z'' + \frac{b^2\,.\,{}'b\,.\,{}''b}{2\,.\,2\,.\,2}\,.\,\frac{(1+{}'b)}{2}\,.\,\frac{(1+{}''b)}{2}\,\frac{(1+{}'''b)}{2}\,z'''\right.$$
$$\left. + \&c.\right\}$$
$$+\frac{b\,.\,{}'P}{2^n}\left\{\frac{b}{2} + \frac{b\,.\,{}'b}{2\,.\,2} + \frac{b\,.\,{}'b\,.\,{}''b}{2\,.\,2\,.\,2} + \&c.\right\}\text{h. log.}\;\zeta + \sqrt{(1+\zeta^2)};$$

since the last term, to wit, $\frac{b^2\,.\,{}'b\,.\,{}''b\,\ldots\ldots\,\beta}{2\,.\,2\,.\,2}\,.\,\frac{(1+{}'b)}{2}\,.\,\frac{(1+{}''b)}{2}\,\ldots\ldots\,\frac{(1+\beta)}{2}\,\times$ $\zeta+\sqrt{(1+\zeta^2)}$, or $\frac{\beta^2}{{}'P}\,.\,\frac{(\zeta+\sqrt{(1+\zeta^2)})}{2}$, may be neglected, on account of its smallness.

Suppose it were required to find, from this form, the whole integral of $dx\sqrt{\left(\frac{1-e^2\,x^2}{1-x^2}\right)}$ from $x=0$ to $x=1$, put $v = \frac{1}{\sqrt{b}}$, then $z' = \frac{2}{1+{}'b}\,.\,v\sqrt{\left(\frac{1+v^2}{1+b^2\,v^2}\right)} = \frac{2}{(1+{}'b)\,.\,b} = \left(\text{since } b^2 = \frac{4\,{}'b}{(1+{}'b)^2}\right)\frac{1}{\sqrt{{}'b}}$; similarly, $z'' = \frac{1}{\sqrt{{}''b}}$, $z''' = \frac{1}{\sqrt{{}'''b}}$ &c.

Consequently, since

$$f = v\sqrt{\left(\frac{1+b^2\,v^2}{1+v^2}\right)}$$
$$-\frac{b^2}{2}\left(\frac{1+{}'b}{2}\,z' + \frac{{}'b}{2}\,.\,\frac{(1+{}'b)}{2}\,.\,\frac{(1+{}''b)}{2}\,z'' + \frac{{}'b\,.\,{}''b}{2\,.\,2}\,.\,\frac{(1+{}'b)}{2}\,.\,\frac{(1+{}''b)}{2}\,.\,\frac{(1+{}'''b)}{2}\,z'''\right.$$
$$\left.+ \&c.\right)$$
$$+\frac{b\,.\,{}'P\,{}'Q}{2^n}\,.\,l.\left(\zeta+\sqrt{(1+\zeta^2)}\right),$$

($l.$ denoting the Naperean logarithm,) when $v = \frac{1}{\sqrt{b}}$,

$$f = 1$$
$$-\frac{b}{2}\left(\frac{b\,.\,1+{}'b}{2}\,.\,\frac{1}{\sqrt{{}'b}} + \frac{b\,{}'b\,(1+{}'b)\,(1+{}''b)}{2\,.\,2\;\;\;.\;\;\;2}\,.\,\frac{1}{\sqrt{{}''b}} + \&c.\right\}$$
$$+\frac{b\,.\,{}'P\,{}'Q}{2^n}\,.\,l.\left(\frac{1+\sqrt{(1+\beta)}}{\sqrt{\beta}}\right)$$

$$= 1$$
$$- \frac{b}{2}\left\{1 + \frac{b \,.\, 1+{}^{\backprime}b}{2 \,.\, 2} + \frac{b \,.\, {}^{\backprime}b \,.\, (1+{}^{\backprime}b)\,(1+{}^{\backprime\backprime}b)}{2 \,.\, 2 \,.\, 2 \,.\, 2} + \&c.\right\}$$
$$+ \frac{b}{2^n} \,.\, {}^{\backprime}P \,.\, {}^{\backprime}Q \,.\, l. \left(\frac{1+\sqrt{(1+\beta)}}{\sqrt{\beta}}\right);$$

but it has been shown that, $f(1)$ denoting the integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ when $x = 1, f(1) = 2f\left(x = \frac{1}{\sqrt{(1+b)}}\right) - 1 + b$; consequently,

$$f(1) = 1 \qquad (8)$$
$$- b\left\{\frac{b \,.\, 1+{}^{\backprime}b}{2 \,.\, 2} + \frac{b \,.\, {}^{\backprime}b \,.\, (1+{}^{\backprime}b)\,(1+{}^{\backprime\backprime}b)}{2 \,.\, 2 \,.\, 2 \,.\, 2} + \&c.\right\}$$
$$+ \frac{2b}{2^n} \,.\, {}^{\backprime}P \,.\, {}^{\backprime}Q \,.\, l. \left(\frac{1+\sqrt{(1+\beta)}}{\sqrt{\beta}}\right).$$

Since ${}^{\backprime}b = \frac{b^2}{(1+\sqrt{(1-b^2)})^2}$, ${}^{\backprime\backprime}b = \frac{{}^{\backprime}b^2}{(1+\sqrt{(1-{}^{\backprime}b^2)})^2}$ &c. the terms ${}^{\backprime}b, {}^{\backprime\backprime}b, {}^{\backprime\backprime\backprime}b$, &c. decrease very rapidly; and b being a small fraction, ${}^{\backprime}b$ is nearly $= \frac{b^2}{4}$, ${}^{\backprime\backprime}b$ more nearly $= \frac{{}^{\backprime}b^2}{4}$, ${}^{\backprime\backprime\backprime}b$ more nearly $= \frac{{}^{\backprime\backprime}b^2}{4}$, &c. Suppose, then, in the series ${}^{\backprime}b, {}^{\backprime\backprime}b, {}^{\backprime\backprime\backprime}b$, &c. β, that $\beta_{\backprime}, \beta_{\backprime\backprime}, \beta_{\backprime\backprime\backprime}$, &c. are the terms preceding β, $\beta = \frac{\beta_{\backprime}^2}{4}$, $\beta_{\backprime} = \frac{\beta_{\backprime\backprime}^2}{4}$, &c. consequently, $l.\left(\frac{1+\sqrt{(1+\beta)}}{\sqrt{\beta}}\right) = l.\frac{2}{\sqrt{\beta}} = \frac{1}{2}\,.\,l.\frac{4}{\beta} = \frac{1}{2}\,.\,l.\frac{4^2}{\beta_{\backprime}^2} = 2\,.\,l.\frac{2}{\sqrt{\beta_{\backprime}}} =$, similarly, $2^2.\,l.\frac{2}{\sqrt{\beta_{\backprime\backprime}}} = 2^3.\,l.\frac{2}{\sqrt{\beta_{\backprime\backprime\backprime}}}$ &c. Hence, supposing ${}^{(m)}b$ the term in the series ${}^{\backprime}b, {}^{\backprime\backprime}b, {}^{\backprime\backprime\backprime}b$, &c.* after which, without sensible error, each term $= \frac{1}{4}$ of the square of the preceding term, we have

$$f(1) = 1$$
$$- b\left\{\frac{b \,.\, 1+{}^{\backprime}b}{2 \,.\, 2} + \frac{b \,.\, {}^{\backprime}b \,.\, (1+{}^{\backprime}b)\,(1+{}^{\backprime\backprime}b)}{2 \,.\, 2 \,.\, 2 \,.\, 2} + \&c.\right\} \qquad (9)$$
$$+ \frac{2b}{2^m} \,.\, {}^{\backprime}P \,.\, {}^{\backprime}Q \,.\, l. \left(\frac{2}{\sqrt{{}^{(m)}b}}\right),$$

* Or thus, $l.\,\frac{1+\sqrt{(1+\beta)}}{\sqrt{\beta}} = l.\frac{2}{\sqrt{\beta}} \,.\, + \frac{\beta}{4} - \frac{1 \,.\, 3}{2 \,.\, 4} \,.\, \frac{\beta^2}{4} + \&c. = l.\frac{2}{\sqrt{\beta}} + \frac{\beta}{4}$, very nearly $\therefore$, instead of $l.\left(\frac{1+\sqrt{1+\beta}}{\sqrt{\beta}}\right)$, we may put $l.\frac{2}{\sqrt{\beta}} + \frac{\beta}{4}$, or $2l.\frac{2}{\sqrt{\beta_{\backprime}}} + \frac{\beta_{\backprime}^2}{4^2}$, or $2^2\,l.\frac{2}{\sqrt{\beta_{\backprime\backprime}}} + \frac{\beta_{\backprime\backprime}^4}{4^4}$, or $2^3.\,l.\frac{2}{\sqrt{\beta_{\backprime\backprime\backprime}}} + \frac{\beta_{\backprime\backprime\backprime}^8}{4^8}$, or $2^{n-m}\,l.\frac{2}{{}^{(m)}b}$, very nearly.

or $= 1$

$$- b\left\{\frac{b \cdot 1 + {}^{\backprime}b}{2 \cdot 2} + \frac{b \cdot {}^{\backprime}b\,(1 + {}^{\backprime}b)\,(1 + {}^{\backprime\backprime}b)}{2 \cdot 2 \cdot 2 \cdot 2} + \&c.\right\} \qquad (9)$$

$$+ \frac{b}{2^m} \cdot {}^{\backprime}P \cdot {}^{\backprime}Q \cdot l. \frac{4}{{}^{(m)}b},$$

in which, the last term, $\frac{b}{2^m} \cdot {}^{\backprime}P \cdot {}^{\backprime}Q \cdot l. \frac{4}{{}^{(m)}b}$, is, in particular values of m, $\frac{b}{2^2} \cdot {}^{\backprime}P \cdot {}^{\backprime}Q \cdot l. \frac{4}{{}^{\backprime\backprime}b}$, or $\frac{b}{2^3} \cdot {}^{\backprime}P \cdot {}^{\backprime}Q \cdot l. \frac{4}{{}^{\backprime\backprime\backprime}b}$, or $\frac{b}{2^4} \cdot {}^{\backprime}P \cdot {}^{\backprime}Q \cdot l. \frac{4}{{}^{\backprime\backprime\backprime\backprime}b}$ &c. each successive value being nearer the truth.

Let $b = \sqrt{\frac{1}{2}} \therefore \sqrt{(1 - b^2)}$ or $e = \sqrt{\frac{1}{2}}$; hence, the two formulas for the integral of $dx\sqrt{\left(\frac{1 - e^2 x^2}{1 - x^2}\right)}$ being equal, and the terms of the series $b, {}^{\backprime}b, {}^{\backprime\backprime}b, {}^{\backprime\backprime\backprime}b$, &c. being respectively equal the terms of the series e, e', e'', e''', &c. we have $P = {}^{\backprime}P$, $Q = {}^{\backprime}Q$, and

$$\frac{P \cdot \pi}{2} \cdot (1 - bQ)$$

$$= 1 - b\left\{\frac{b \cdot 1 + {}^{\backprime}b}{2 \cdot 2} + \frac{b \cdot {}^{\backprime}b \cdot (1 + {}^{\backprime}b)\,(1 + {}^{\backprime\backprime}b)}{2 \cdot 2 \cdot 2 \cdot 2} + \&c.\right\} + \frac{b}{2^m} \cdot P \cdot Q \cdot l. \frac{4}{{}^{(m)}b}.$$

The two forms (5) (7), are fully adequate to the computation of the integral of $dx\sqrt{\left(\frac{1 - e^2 x^2}{1 - x^2}\right)}$ in all values of e; the series (5), involving e, e', e'', e''', &c. is to be used, when e is any value between o and $\sqrt{\frac{1}{2}}$; and the series (7), involving $b \cdot {}^{\backprime}b, {}^{\backprime\backprime}b, {}^{\backprime\backprime\backprime}b$, &c. is to be used, when e is any value between $\sqrt{\frac{1}{2}}$ and 1, or, what is the same thing, when b is any value between o and $\sqrt{\frac{1}{2}}$.

From the preceding forms may be deduced a very curious and remarkable theorem for the circumference of a circle, which I shall now exhibit.

By former substitution, $u' = \frac{2}{1 + e'}\, x \sqrt{\left(\frac{1 - x^2}{1 - e^2 x^2}\right)}$;

and $\therefore$, when $x = \frac{1}{\sqrt{(1 + b)}}$, $u' = 1$.

Hence, $\int \frac{dx}{\sqrt{(1 - x^2)\,(1 - e^2 x^2)}} \left(x = \frac{1}{\sqrt{(1 + b)}}\right) = \frac{1 + e'}{2} \int \frac{du'}{U'}\ (u' = 1)$,

and $\int \frac{dx}{\sqrt{(1 - x^2)\,(1 - e^2 x^2)}}\ (x = 1) = \frac{1 + e'}{2} \cdot 2 \int \frac{du'}{U'}\ (u' = 1)$.

Consequently,

$$\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}\,(x=1)=2\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}\,\left(x=\frac{1}{\sqrt{(1+b)}}\right);$$

but $\displaystyle\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}=\frac{(1+e')}{2}\cdot\frac{(1+e'')}{2}\cdots\cdots\frac{1+\varepsilon}{2}\int\frac{dv}{\sqrt{(1-v^2)}}$

$$=(1+e')(1+e'')\ 1+\varepsilon\,.\,\frac{\pi}{2},\text{ when } x=1.$$

Again, $\displaystyle\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}=\int\frac{dv}{\sqrt{(1+v^2)(1+b^2v^2)}}$, putting $x=\dfrac{v}{\sqrt{1+v^2}}$,

and $\displaystyle\int\frac{dv}{\sqrt{(1+v^2)(1+b^2v^2)}}=\frac{1+{}'b}{2}\cdot\frac{1+{}''b}{2}\cdots\frac{1+\beta}{2}\,l.\,(\zeta+\sqrt{1+\zeta^2})$,

and, when $v=\dfrac{1}{\sqrt{b}}$, that is, when $x=\dfrac{1}{\sqrt{(1+b)}}$,

$$\int\frac{dv}{\sqrt{(1+v^2)(1+b^2v^2)}}=\frac{1+{}'b}{2}\cdot\frac{1+{}''b}{2}\cdots\frac{1+\beta}{2}\cdot l.\left(\frac{1+\sqrt{(1+\beta)}}{\sqrt{\beta}}\right).$$

Hence, since $\displaystyle\int\frac{dv}{\sqrt{(1+v^2)(1+b^2v^2)}}\left(v=\frac{1}{\sqrt{b}}\right)=\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$,

$\displaystyle\left(x=\frac{1}{\sqrt{(1+b)}}\right)=\frac{1}{2}\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}\,(x=1)$,

we have $2\,.\left\{\dfrac{(1+{}'b)}{2}\cdot\dfrac{(1+{}''b)}{2}\cdots\dfrac{1+\beta}{2}\right\}l.\left(\dfrac{1+\sqrt{(1+\beta)}}{\sqrt{\beta}}\right)$

$=(1+e')(1+e'')\ \ldots\ (1+\varepsilon)\,.\,\dfrac{\pi}{2}$.

Let now $e=\sqrt{\frac{1}{2}}\ \therefore\ b=\sqrt{\frac{1}{2}}\ \therefore\ e'={}'b,\ e''={}''b$ &c. and $\varepsilon=\beta$

$$\therefore\ \frac{2\,.\,l.\,(1+\sqrt{(1+\beta)}}{2^n\times\sqrt{\beta}}=\frac{\pi}{2};$$

or, from what has preceded,

$$\frac{l.\,\frac{4}{{}^{(m)}b}}{2^m}=\frac{\pi}{2}.$$

In particular cases,

$\dfrac{\pi}{2}=2^{-3}.\,l.\,\dfrac{4}{{}'''b}=$ (more nearly) $2^{-4}.\,l.\,\dfrac{4}{{}^{\text{iv}}b}$ (more nearly) $2^{-5}.\,l.\,\dfrac{4}{{}^{\text{v}}b}$ &c.*

* Or thus, when $e=\sqrt{\frac{1}{2}}$. e^{iv} will be a very small fraction, for 10 zeros will precede the first significant figure.

$(1+e^{\text{v}})(1+e^{\text{vi}})(1+e^{\text{vii}})\ \ldots\ (1+\varepsilon)\,\dfrac{\pi}{2}=22^{-5}.\,l.\left(\dfrac{1+\sqrt{1+{}^{\text{v}}b}}{\sqrt{{}^{\text{v}}b}}\right)$,

or $(1+e^{\text{v}})(1+e^{\text{vi}})(1+e^{\text{vii}})\ldots(1+\varepsilon)\,.\,\dfrac{\pi}{2}=2^{-4}\,.\,l.\,\dfrac{4}{{}^{\text{iv}}b}$,

or very nearly $\dfrac{\pi}{2}=2^{-4}\,.\,l.\,\dfrac{4}{{}^{\text{iv}}b}$.

It has been shown, (page 239), that $\int\frac{(A+Bx^2)\,dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$ (F) may be transformed into a form such as

$$m\,.\,u'+n\int\frac{du'}{U'}+\alpha.\int\frac{(A'+B'u'^2)\,du'}{\sqrt{(1-u'^2)(1-e'^2u'^2)}}\ (F');$$

and, similarly, F′ into a form as $m'u''+n'\int\frac{du''}{U''}+\alpha'.F''$.

Consequently, since $\int\frac{du'}{U'}=\frac{1+e''}{2}.\int\frac{du''}{U''}$, we can exterminate $\int\frac{du''}{U''}$, and obtain a resulting equation, such as $\beta F+\gamma F'+\delta F''+pu'+qu''=0$; which expresses the relation between the integrals of three expressions similar to $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$.

If $x=1$, then u', u'', u''', &c. $=0$; consequently, $\beta F(1)+2\gamma F'(1)+4\delta.F''(1)=0$, since, x passing from 0 to 1, u' passes from 0 to its maximum (1), and from 1 to 0; consequently, between the values of x, 0, and 1, $\int dF'=2F'(1)$, $F'(1)$ representing what the integral F′ becomes when $u'=1$; similarly, $\int dF''$, when $x=1$, $=4F''(1)$.

Since similar equations must be true for F′, F″, F‴, for F″, F‴, F^{iv}, &c. as for F, F′, F″, it is plain that, by a simple process of elimination, we may arrive at an equation of the form $\beta F+\mu F^{(n-1)}+\nu F^{(n)}+\pi u'+\varrho u''+$&c.$=0$, β, π, ν, ϱ, &c. being constant quantities, $F^{(n-1)}$, $F^{(n)}$, the two last terms of the series F′, F″, F‴, &c.

It is clear also, that we can obtain an equation as $\beta F+\gamma F'+\delta F''+\varepsilon F'''+$&c. .... $\mu F^{(n-1)}+\nu F^{(n)}+nu'+\varrho u''+\sigma u'''+$&c. ... $=0$.

If, in particular applications, $\int\frac{(A+Bx^2)\,.\,dx}{\sqrt{(1-e^2x^2)(1-x^2)}}$ represents the arc or area of a curve, the foregoing results, differently expressed, will announce properties subsisting between the arcs and areas of similar curves; for instance, when $A=1$, $B=e^2$, the integral $\int\frac{(1-e^2x^2)\,dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$ expresses the arc of an ellipse, abscissa x, semi-

axes 1 and $\sqrt{(1-e^2)}$; consequently, the arc of one ellipse may be represented by the arcs of other ellipses whose excentricities vary according to a given law; thus,

$\frac{1-e'}{2}\cdot\int\frac{du'}{U'}$, or $\frac{(1-e')}{2}\cdot\frac{(1+e'')}{2}\int\frac{du''}{U''} = \frac{e^2}{4}\cdot(1+e')\,u' - f + \frac{f'}{1+e'}$,

and $\frac{1-e''}{2}\int\frac{du''}{U''} = \frac{e'^2}{4}(1+e'')\,u'' - f' + \frac{f''}{1+e''}$.

Consequently,

$e'u' - (1+e')f + f' = \frac{1+e''}{2}\cdot\frac{1-e'^2}{1-e''^2}\{e''u'' - f'(1+e'') + f''\}$,

or $e'u' - (1+e')f + \frac{3+e''}{2\cdot(1+e'')}\cdot f' - \frac{1-e''}{2(1+e'')^2}f'' - \frac{e''(1-e'')}{2\cdot(1+e'')^2}u'' = 0$;

which equation, calling u' sin. $2\theta'$, u'' sin. $4\theta''$, agrees with the equation

$2\cdot(1+c')\,E'' = \frac{3+c}{1+c}E' - \frac{1-c}{(1+c)^2}(E + c\cdot\sin.\varphi) + 2c\cdot\sin.\varphi$,

given by LEGENDRE, *Mém. de l'Academie*, 1786, page 657.

If $x = 1$, $u' = 0$, and $u'' = 0$; consequently,

$$2(1+e')f(1) - \frac{3+e''}{(1+e'')}\cdot 2f'(1) + \frac{(1-e'')}{(1+e'')^2}\cdot 4f''(1) = 0. \qquad (b)$$

Putting $x = \frac{v}{\sqrt{(1+v^2)}}$, we have (see page 246) $df = dv\frac{\sqrt{(1+b^2v^2)}}{(1+v^2)^{\frac{3}{2}}}$

$= d\left\{v\sqrt{\left(\frac{1+b^2v^2}{1+v^2}\right)}\right\} - \frac{b^2}{4}(1+{}'b)\,dz' + \frac{b^2}{4}(1+{}'b)\frac{dz'}{Z'} - \frac{{}'b^2}{1+{}'b}\cdot\frac{z'^2\,dz'}{Z'}$,

or $f = v\sqrt{\left(\frac{1+{}'b^2v^2}{1+v^2}\right)} - \frac{b^2}{4}(1+{}'b)\,z' + \frac{b^2}{4}\cdot(1+{}'b)\frac{1+{}''b}{2}\cdot\frac{dz''}{Z''} +$

$\frac{1}{1+{}'b}\,{}'f - \frac{1}{1+{}'b}\cdot z'\sqrt{\left(\frac{1+{}'b^2z'^2}{1+z'^2}\right)}$.

Similarly, ${}'f = z'\sqrt{\left(\frac{1+{}'b^2z'^2}{1+z'^2}\right)} - \frac{{}'b^2}{4}(1+{}''b)\,z'' + \frac{{}'b^2}{4}(1+{}''b)\cdot\frac{dz''}{Z''} +$

$\frac{1}{1+{}''b}\cdot{}''f - \frac{1}{1+{}''b}\cdot z''\sqrt{\left(\frac{1+{}''b^2z''^2}{1+z''^2}\right)}$.

Hence, exterminating $\frac{dz''}{Z''}$, there results an equation between f, ${}'f$, ${}''f$, and certain functions of v.

If $v = \frac{1}{\sqrt{b}}$, $z' = \frac{1}{\sqrt{{}'b}}$, $z'' = \frac{1}{\sqrt{{}''b}}$, substitute these quantities, and

for f, $'f$, $''f$, put $\frac{f(1)+1-b}{2}$, $\frac{'f(1)+1-'b}{2}$, $\frac{''f(1)+1-''b}{2}$, and there results an equation between $f(1)$, $'f(1)$, $''f(1)$, the same as the one given in the preceding page.

LANDEN, Mem. page 35, and LEGENDRE, *Mém. de l'Acad.* 1786, p. 678, have deduced an equation subsisting between the circumference of a circle and the peripheries of two ellipses, whose excentricities are $\sqrt{\frac{1}{2}}$ and $\frac{\sqrt{(2\sqrt{2})}}{1+\frac{1}{2}\sqrt{2}}$; * but the application of the preceding forms will enable us to express, immediately, the relation between the peripheries of a circle and of two ellipses, the excentricity of one ellipse being assumed of any magnitude; thus, by equation (a), page 242.

$$\int \frac{du'}{U'} = \frac{2}{1+e'}\left\{\frac{e^2 u'}{1+e'} - f + \frac{f'}{1+e'}\right\};\ \text{consequently, since}$$

$$\int \frac{dx}{\sqrt{(1-x^2)(1-e^2 x^2)}} = \frac{1+e'}{2} \cdot \int \frac{du'}{U'},\ \text{when } x = 1,$$

$$\int \frac{dx}{\sqrt{(1-x^2)(1-e^2 x^2)}} = \frac{2f'(1)}{1-e'} - \frac{1+e'}{1-e'} \cdot f(1);$$

$$\text{but} \int \frac{dx}{\sqrt{(1-x^2)(1-e^2 x^2)}}\ (x=1) = \mathrm{P} \cdot \frac{\pi}{2},\ (\text{page } 250)$$

$$\therefore \mathrm{P} \cdot \frac{\pi}{2} = \frac{2f'(1)}{1-e'} - \frac{1+e'}{1-e'} f(1);$$

or, $(1-e') \cdot \mathrm{P} \cdot \pi - 4f'(1) + 2 \cdot (1+e') f(1) = 0$;

or, since $\frac{\pi}{2}$ = quadrant of circle (q) radius $= 1$,

$$(1-e')\,\mathrm{P} \cdot q - 2f'(1) + (1+e') f(1) = 0.$$

And f, f', may represent arcs of ellipses described on the same semiaxis, major (1), with excentricities equal to e, e' being $= \frac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}}$.

* The semiaxes of the two ellipses compared by LANDEN, are $\sqrt{2}$, 1, and $\frac{1}{\sqrt{2}} + \frac{1}{2}$, $\frac{1}{\sqrt{2}} - \frac{1}{2}$.

It is plain that, by a similar method, we may deduce an equation between $f(1)$, $f'(1)$, and $l.\frac{4}{{}^{(m)}b}$.

LEGENDRE puts $c'' = \frac{2\sqrt{c'}}{1+c'}$; therefore f answers to E'' in his equation.

From what has preceded it appears, that the forms

$$f = \frac{e^2}{4}(1+e')u' + \frac{e^2}{4}\cdot\frac{e'(1+e')(1+e'')}{4}u'' + \&c. + \frac{P(1-eQ)}{2^n}\int\frac{dv}{V},$$

$$f = v\sqrt{\left(\frac{1+b^2v^2}{1+v^2}\right)}$$

$$-\left\{\frac{b^2}{4}\cdot(1+{}'b)z' + \frac{b^2\,{}'b\,(1+{}'b)(1+{}''b)}{4\cdot 4}z'' + \&c.\right\}$$

$$+\frac{b\cdot{}'P\,{}'Q}{2^n}\cdot l.(\zeta+\sqrt{(1+\zeta^2)}),$$

$$f = \frac{e'u'}{1+e'} - \frac{e''(1-e'')\cdot u''}{2\cdot(1+e')(1+e'')^2} + \frac{3+e''}{2\cdot(1+e')(1+e'')}\cdot f' - \frac{(1-e'')f''}{2\cdot(1+e')(1+e'')^2}\ \&c.$$

are parts of the same method of computation, differently expressed. It also appears, how certain analytical artifices of computation, translated into geometrical language, become curious properties of curves.

FAGNANI's theorem, as it is called, may be deduced from the form for the transformation of f; thus, taking the simplest case,

$$df = \frac{e^2}{4}(1+e')\cdot du' - \frac{(1-e')}{2}\cdot\frac{du'}{U'} + \frac{1}{1+e'}\cdot df',$$

when u' is at its maximum, (1) $x = \frac{1}{\sqrt{(1+b)}}$

$$\therefore f\left(x = \frac{1}{\sqrt{(1+b)}}\right) = \frac{e^2}{4}\cdot(1+e') - \frac{1-e'}{2}\cdot\int\frac{du'}{U'}(u'=1) + \frac{f'(1)}{(1+e')}$$

$(u'=1)$,

and $f(1)\ (x=1) = \qquad -\frac{1-e'}{2}\cdot 2\int\frac{du'}{U'} + \frac{2f'(1)}{1+e'}.$

Consequently, $2f - f(1) = \frac{2e^2}{4}(1+e') = (1-b^2)\cdot\frac{1+e'}{2}$

$= (1-b^2)\frac{1}{1+b} = 1-b;$

or $2f = f(1) + 1 - b$, or $f - \{f(1) - f\} = 1 - b$,
or $f - \frac{f(1)}{2} = \frac{1-b}{2}$.

This, expressed with reference to an ellipse, announces that the difference between an arc of an ellipse $\left(\text{abscissa} = \frac{1}{\sqrt{(1+b)}}\right)$ and half the quadrant of an ellipse, equals half the difference of the semiaxes.

Similarly, the difference between f and $\frac{f(1)}{4}$ may be asssigned, when $f = \int dx \sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, $u'' = 1$, and, consequently, when $u' = \frac{1}{\sqrt{(1+b')}}$ $= \frac{2}{1+e'} x \sqrt{\left(\frac{1-x^2}{1-e^2x^2}\right)}$; thus, supposing the value of x to be a, when $u''=1$; since,

$$df = \frac{e'}{1+e'} \cdot du' + \frac{e''}{(1+e')(1+e'')} \cdot du'' - \left\{\frac{1-e'}{2} \cdot \frac{1+e''}{2} + \frac{1-e''}{2.(1+e')}\right\} \frac{du''}{U''}$$
$$+ \frac{df''}{(1+e')(1+e'')},$$

$$\therefore f(x=a) = \frac{e'}{1+e'} \cdot \frac{1}{\sqrt{(1+b')}} + \frac{e''}{(1+e')(1+e'')} - \left\{\frac{(1-e')(1+e'')}{2 \,.\, 2} + \frac{1-e''}{2.(1+e')}\right\} \int \frac{du''}{U''} + \frac{f''}{(1+e')(1+e'')},$$

and $f(1) = -\left\{\frac{(1-e')(1+e'')}{2 \,.\, 2} + \frac{(1-e'')}{2.(1+e')} \cdot\right\} 4\int \frac{du''}{U''} + \frac{4f''(1)}{(1+e')(1+e'')}$,

$$\therefore 4f(x=a) - f(1) = \frac{4e'}{1+e'} \cdot \frac{1}{\sqrt{(1+b')}} + \frac{4e''}{(1+e')(1+e'')}.$$

Now, $1+b' = \frac{(1+\sqrt{b})^2}{1+b}$, $e' = \frac{1-b}{1+b}$, and $e'' = \frac{1-b'}{1+b'}$;
consequently,

$$4f - f(1) = 2(1-\sqrt{b})\sqrt{(1+b)} + (1-\sqrt{b})^2,$$
or $f - \frac{1}{4}f(1) = \frac{(1-\sqrt{b})\sqrt{(1+b)}}{2} + \left(\frac{1-\sqrt{b}}{2}\right)^2$.

Now, to determine x, we have $u' = \sqrt{\left(\frac{1+e''}{2}\right)} = \frac{2}{1+e'} . x \sqrt{\left(\frac{1-x^2}{1-e^2x^2}\right)}$;
consequently, putting $\sqrt{\left(\frac{1+e''}{2}\right)} = m$,

$x^2 = \frac{1+m^2e' - \sqrt{(1-m^2)(1-m^2e'^2)}}{}$, or, putting for m^2, e', their values

$$x^2 = \frac{1}{1+\sqrt{b}}\left\{1 - \frac{\sqrt{b}}{\sqrt{(1+b)}}\right\};$$

which conclusion agrees with LEGENDRE's, obtained by a different process. See *Mem. de l'Academie*, 1786, p. 665,

The foregoing method may be continued at pleasure; thus, if $u'''=1$, then $u''=\sqrt{\left(\frac{1+e'''}{2}\right)}$, and $u''^2 = \frac{1}{1+\sqrt{b'}}\left\{1-\frac{\sqrt{b'}}{\sqrt{(1+b')}}\right\}$; and, putting this value $= m'^2$, x^2 must be determined from the equation $x^2 = \frac{1+m'^2 e' - \sqrt{(1-m'^2)(1-m'^2 e'^2)}}{2}$; and, similarly must the process be conducted, if u^{iv}, or u^{v}, or $u^{\text{vi}}=1$.

These results, applied to an ellipse, cause it to appear, that right lines can be assigned, respectively equal to the difference between an arc and half the quadrant, between an arc and one-fourth of the quadrant, between an arc and one-eighth of the quadrant, &c.

Here may again be remarked, the connexion between the artifices of computation and the properties of curves; for the series expressing $\int dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ (*cæteris paribus*) converges more quickly, the less x is; consequently, the whole integral is more commodiously calculated by the theorem $f(1) = 2f\left(x=\frac{1}{\sqrt{(1+b)}}\right)$ $-1+b$, than if x were put $=1$, in the form of the expansion of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$; still more commodiously, by the theorem

$$f(1) = 4f(x=a) - 2(1-\sqrt{b})\sqrt{(1+b)} - (1-\sqrt{b})^2,$$

where $a^2 = \frac{1}{1+\sqrt{b}}\left\{1-\frac{\sqrt{b}}{\sqrt{(1+b)}}\right\}$ is less than $\frac{1}{\sqrt{(1+b)}}$, and so on.

It has been already observed, that the methods of determining f, by f', and f'', or by f', f'', f''', or by f'', f''', &c. as LEGENDRE has done, or by the regular form which the indefinite reduction of f, into $f^{(n-1)}, f^{(n)}$, assumes, are, *au fond*, the same methods; and I purpose now to show that the substitution, which is to be considered as the base and principle of the method, is the same, although dif-

ferently expressed, in the methods of LEGENDRE, of Mr. IVORY, and of Mr. WALLACE, who have learnedly and ingeniously written on this subject.

In order to deduce the relation between three ellipses, LEGENDRE, *Mem. de l'Academie*, 1786, p. 650, assumes

$$(1-b')\ \sin.\ \phi = \frac{c'^2.\ \sin.\ \phi'.\ \cos.\ \phi'}{\sqrt{(1-c^2\ \sin.\ \phi')}}.$$

Now, according to this author's notation, $c'^2 = \frac{4c}{(1+c)^2}$, and $1-b' = \frac{2c}{1+c}$; consequently, $\sin.\ \phi = \frac{2}{1+c}\cdot\frac{\sin.\ \phi'.\ \cos.\ \phi'}{\sqrt{(1-c'^2.\ \sin.\ ^2\phi')}}$, which is precisely the same substitution as $u' = \frac{2}{1+e'}\ x \sqrt{\left(\frac{1-x^2}{1-e^2\ x^2}\right)}$.

In the Edinb. Trans. Vol. IV. p. 183, $\sin.\ (\psi-\phi)$ is assumed $= c\ .\ \sin.\ \psi$; but $\sin.\ (\psi-\phi) = \sin.\ \psi\ .\ \cos.\ \phi - \cos.\ \psi\ .\ \sin.\ \phi$,

$$\therefore \sin.\ ^2\psi = \frac{\sin.\ ^2\phi}{1+c^2-2c\ .\ \cos.\ \phi} = \frac{\left(2\ .\ \sin.\ \frac{\phi}{2}\ .\ \cos.\ \frac{\phi}{2}\right)^2}{1+c^2-2c\ (2\ (\cos.\ \frac{\phi}{2})^2-1)};$$

consequently, putting $\frac{4c}{(1+c)^2} = e^2$,

$$\sin.\ \psi = \frac{2\ .\ \sin.\ \frac{\phi}{2}\ .\ \cos.\ \frac{\phi}{2}}{(1+c)\sqrt{(1-e^2.(\cos.\ \frac{\phi}{2})^2)}},$$ the same substitution as

$$u' = \frac{2}{1+e'}\ .\ x\sqrt{\left(\frac{1-x^2}{1-e^2\ x^2}\right)}.$$

Again, in Edinb. Trans. Vol. V. p. 272, $\sin.\ 2\phi'$ is made $= \frac{\sin.\ 2\phi}{\sqrt{(1+e'^2+2e'\ .\ \cos.\ \phi)}} =$, consequently, $\frac{2\ .\ \sin.\ \phi\ .\ \cos.\ \phi}{\sqrt{1+e'^2+2e'\ (1-2\ \sin.\ ^2\phi)}} = \frac{2}{1+e'}\cdot\frac{\sin.\ \phi\ .\ \cos.\ \phi}{\sqrt{(1-e^2\ \sin.\ ^2\phi)}}$ $\left(e^2 = \frac{4e'}{(1+e')^2}\right)$, the same substitution as $u' = \frac{2}{1+e'}.x\sqrt{\left(\frac{1-x^2}{1-e^2\ x^2}\right)}$.

It appears, then, that the preceding substitutions, although, by the aid of geometrical language, differently expressed, are all reducible to the algebraical substitution of $u' = \frac{2}{1+e'}.x\sqrt{\left(\frac{1-x^2}{1-e^2\ x^2}\right)}$, in the

form $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$; which substitution I conceive to be more obvious, more easily suggested, and more analogous to ordinary algebraical substitutions, than the substitution of $\frac{\sin. 2\phi}{\sqrt{(1+e'^2+2e'.\cos\phi)}}$ for the sin. $2\phi'$, or, of $\frac{\sin.(\psi-\phi)}{c}$ for sin. ψ.

Of this substitution of u' for $x\sqrt{\left(\frac{g+hx^2}{e+fx^2}\right)}$, and of the transformation of $dx\sqrt{\left(\frac{e+fx^2}{g+hx^2}\right)}$ into $Adu'+dx'\sqrt{\left(\frac{e'+f'x'^2}{g'+h'x^2}\right)}$, &c. M. LAGRANGE is, I believe, the original author.

When x is called the sine of an arc θ, $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ may be expressed by $d\theta\sqrt{(1-e^2\sin.^2\theta)}$, $\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, by $\frac{d\theta}{\sqrt{(1-e^2\sin.^2\theta)}}$. LAGRANGE, *Fonct. Analyt.* p. 90, has treated of the integrals of these expressions; as has LEGENDRE, *Mem. de l'Acad.* p. 663, and LACROIX, *Traité du Calcul diff.* Vol. II. page 454.

The results obtained by these authors, may easily be deduced from the substitution of $x\sqrt{\left(\frac{1-x^2}{1-e^2x^2}\right)}=v\sqrt{\left(\frac{1-v^2}{1-e^2v^2}\right)}$. Some of these results may appear curious; but I apprehend, what is chiefly necessary for the solutions of problems in physics and astronomy, into which the expressions $\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ enter, is a method of approximating to their integrals.

A certain method of approximating to these integrals, has been given in the preceding pages. In different applications, its expression may be varied; thus, f is transformed into an expression involving f', f'', where f', f'', can be more easily computed than f; express this transformation with reference to an ellipse, and it appears that the length of one ellipse may be estimated, from the lengths of two ellipses of different excentricity. Again, $\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, in order to be computed, is transformed into

$\frac{1+e'}{2} \cdot \int \frac{du}{\sqrt{(1-u'^2)\,(1-e^2 u'^2)}}$, or into $\frac{(1+e')\,(1+e'')}{2\;.\;2} \cdot \int \frac{du''}{\sqrt{(1-u''^2)\,(1-e''^2 u''^2)}}$, &c. but $\int \frac{m\,dx}{\sqrt{(1-x^2)\,(1-e^2 x^2)}}$ (m a constant quantity) expresses the time of vibration of a pendulum in a circular arc; consequently, the time of vibration of one pendulum may be estimated from the time of vibration of another pendulum, vibrating in a different arc; and, generally, corresponding to relations established between abstract quantities f, f', f'', &c. will be found properties subsisting between those subjects, of which, in particular applications, f, f', f'', &c. become the exponents and expressions.

A certain method for computing the integral of $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$ (df) being obtained, in a systematic treatise, the next business of the analyst would be, to show what differential forms depended for their integration on that of df. Such differential forms are many; and, by the introduction of geometrical language, with considerable embarrassment to the computist, varied in their expression. $\frac{dx}{\sqrt{(1-x^2)\,(1-e^2 x^2)^{\frac{3}{2}}}}$, $\frac{dx}{\sqrt{(1-x^2)\,.\,(1-e^2 x^2)^{\frac{2m+1}{2}}}}$, $d\theta \,.\, \sqrt{(1+m\,.\cos.\,\theta)}$, $\cos.\,n\theta.d\theta.\sqrt{(1+m\,.\cos.\,\theta)}$, $dx \sqrt{\left(\frac{e^2 x^2-1}{x^2-1}\right)}$. ($e$, x greater than 1) may be reduced to depend for their integration, on $\int \frac{dx}{\sqrt{(1-x^2)(2-e^2 x^2)}}$, and $\int dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, ($e$, x less than 1). Amongst these, $dx \sqrt{\left(\frac{e^2 x^2-1}{x^2-1}\right)}$ merits some attention. In an analytical point of view, there is nothing curious or remarkable in the reduction of such a form to $dx \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, and other quantities that can be integrated; but, with certain conditions, $\int dx \sqrt{\left(\frac{e^2 x^2-1}{x^2-1}\right)}$ represents the arc of an hyperbola; consequently, announcing the analytical result in geometrical language, the hyperbola may be

rectified by means of an ellipse; which property is to be reckoned curious, I conceive, because the ellipse and hyperbola are sections of the same solid cone; for, otherwise, I do not perceive why it is more curious, that an hyperbola should be rectified by means of an ellipse, than that any other curve, whose arc $=$ F, (F an integral dependent on $\int dx \surd\left(\frac{1-e^2x^2}{1-x^2}\right)$) should be rectified by means of an ellipse.

In order to integrate $dx \surd\left(\frac{e^2x^2-1}{x^2-1}\right)$ by means of $\int dy \surd\left(\frac{1-e^2y^2}{1-y^2}\right)$, put $x = \surd\left(\frac{1-\frac{z^2}{e^2}}{1-z^2}\right)$; then, when $z=0$ $x=1$, and when $z=1$ $x=\infty$, and $dx\surd\left(\frac{e^2x^2-1}{x^2-1}\right)=\frac{(1-m^2)\,dz}{(1-z^2)^{\frac{3}{2}}\surd(1-m^2z^2)}$, $\left\{\text{putting } m=\frac{1}{e}\right\}$; consequently, $\int dx\surd\left(\frac{e^2x^2-1}{x^2-1}\right)$ $\left(e>1\right\}$ between the values of $x=0$ and $x=\infty$ $=$ integral of $\frac{(1-m^2)\,dz}{(1-z^2)^{\frac{3}{2}}\surd(1-m^2z^2)}$ between the values of $z=0$ and $z=1$.

Now, $d\left\{z\surd\left(\frac{1-m^2z^2}{1-z^2}\right)\right\}=dz\surd\left(\frac{1-m^2z^2}{1-z^2}\right)-\frac{(1-m^2)\,dz}{\surd(1-z^2)(1-m^2z^2)}+\frac{(1-m^2)\,dz}{(1-z^2)^{\frac{3}{2}}(1-m^2z^2)^{\frac{1}{2}}}$.

Hence, $\int\frac{(1-m^2)\,dz}{(1-z^2)^{\frac{3}{2}}\surd(1-m^2z^2)}=z\surd\left(\frac{1-m^2z^2}{1-z^2}\right)-\int dz\surd\left(\frac{1-m^2z^2}{1-z^2}\right)+(1-m^2)\int\frac{dz}{\surd(1-z^2)(1-m^2z^2)}$;

but, if we put $\int dz\surd\left(\frac{1-m^2z^2}{1-z^2}\right)=\mathrm{F}$, $\int d'z\surd\left(\frac{1-'m^2\,'z^2}{1-'z^2}\right)='\mathrm{F}$,

$m=\frac{1-\surd(1-'m^2)}{1+\surd(1-'m^2)}$, $z=\frac{2}{1-m}\cdot{}'z\surd\left(\frac{1-'z^2}{1-'m^2\,'z^2}\right)$.

Then, by equation (a) page 240,

$$\int\frac{dz}{(1-z^2)(1-m^2z^2)}=\frac{2mz}{1-m^2}-\frac{2'\mathrm{F}}{1-m}+\frac{2\mathrm{F}}{1-m^2};$$

consequently, $\int dx\surd\left(\frac{e^2x^2-1}{(x^2-1}\right)=\int\frac{(1-m^2)\,dz}{(1-z^2)^{\frac{3}{2}}\surd(1-m^2z^2)}$

$$= z\sqrt{\left(\frac{1-m^2 z^2}{1-z^2}\right)} + 2mz - 2(1+m)\,'F + F, \qquad (c)$$

which, in fact, is LANDEN's theorem; for $\int dx\sqrt{\left(\frac{e^2 x^2 - 1}{x^2 - 1}\right)}$ represents the arc of an hyperbola, semiaxes 1 and $\sqrt{(e^2-1)}$, and F, F, the arcs of two ellipses.

In an analytical point of view, the latter part of this solution is unnecessary; for the problem is completely resolved, when it is proved that

$$\int\frac{(1-m^2)\,dz}{(1-z^2)^{\frac{3}{2}}\sqrt{(1-m^2 z^2)}} = z\sqrt{\left(\frac{1-m^2 z^2}{1-z^2}\right)} - \int dz\sqrt{\left(\frac{1-m^2 z^2}{1-z^2}\right)} + (1-m^2)\int\frac{dz}{\sqrt{(1-z^2)(1-m^2 z^2)}}.$$

If the differential of $x\sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)}$ be taken, it appears that

$$\int\frac{dx}{\sqrt{(1-x^2)}\,(1-e^2 x^2)^{\frac{3}{2}}} = \frac{1}{1-e^2}\int dx\sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)} - \frac{e^2}{1-e^2}\cdot x\sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)};$$

and hence may be deduced a differential equation of the second order, similar to the one given in page 236. For, since $\frac{df}{dx} = \sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, making e only to vary, or taking the partial differentials, $\frac{d^2 f}{dx\,.\,de} = \frac{-ex^2}{\sqrt{(1-x^2)(1-e^2 x^2)}}$, and $\therefore\ \frac{df}{dx} - \frac{e\,.\,d^2 f}{dx\,.\,de} = \frac{1}{\sqrt{(1-x^2)(1-e^2 x^2)}}$, and $\int\frac{dx}{\sqrt{(1-x^2)(1-e^2 x^2)}} = f - e\,.\,\frac{df}{de}$.

Similarly, $\int\frac{dx}{\sqrt{(1-x^2)}\,(1-e^2 x^2)^{\frac{3}{2}}} = f - e\,.\,\frac{df}{de} - e^2\,.\,\frac{d^2 f}{de^2}$,

or $\frac{f}{1-e^2} - \frac{e^2}{1-e^2}\,x\sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)} = f - e\,.\,\frac{df}{de} - e^2\,.\,\frac{d^2 f}{de^2}$,

or $\frac{e^2 f}{1-e^2} + e\,.\,\frac{df}{de} + e^2\,.\,\frac{d^2 f}{de^2} - \frac{e^2}{1-e^2}\,x\sqrt{\left(\frac{1-x^2}{1-e^2 x^2}\right)} = 0$,

when $x = 1$,

$$\frac{e^2}{1-e^2}\,.\,f(1) + e\,\frac{d f(1)}{de} + \frac{e^2\,d^2 f(1)}{de^2} = 0.$$

I now purpose to show that the integration of forms such as

$\frac{dx}{\sqrt{(1-x^2)}}(1-e^2x^2)^{\frac{\quad}{2}}$, depends on that of $\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, and of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$.

Let $\frac{dx}{\sqrt{1-x^2}}=d\theta$., $1-e^2x^2=R^2$ $\therefore x^2=\frac{1-R^2}{e^2}$, $1-2x^2=\frac{e^2-2+2R^2}{e^2}$; consequently,

$$d\left\{x\sqrt{(1-x^2)}\,R^{2m-1}\right\}=$$

$$d\theta\left\{\frac{e^2-2+2R^2}{e^2}\right\}R^{2m-1}-(2m-1)\,d\theta\;R^{2m-3}\left\{\frac{(1-R^2)(e^2-1+R^2)}{e^2}\right\}$$

$$=2m.\left(\frac{e^2-2}{e^2}\right).R^{2m-1}d\theta+\frac{(2m-1)(1-e^2)}{e^2}.R^{2m-3}\,d\theta+\frac{2m+1}{e^2}.R^{2m+1}d\theta,$$

and $\int R^{2m+1}d\theta=\frac{e^2}{2m+1}.x\sqrt{(1-x^2)}.R^{2m-1}-\frac{2m}{2m+1}(e^2-2).\int R^{2m-1}.d\theta$

$$-\frac{(2m-1)}{2m+1}(1-e^2)\int R^{2m-3}.d\theta; \qquad (d)$$

and, if $(2m+1)$ be negative, either by substitution, or by taking the differential of $\frac{x\sqrt{(1-x^2)}}{R^{2m-1}}$, we have

$$\int\frac{d\theta}{R^{2m+1}}=\frac{2m-2}{2m-1}.\frac{2-e^2}{1-e^2}.\int\frac{d\theta}{R^{2m-1}}-\frac{2m-3}{2m-1}.\frac{1}{1-e^2}.\int\frac{d\theta}{R^{2m-3}}$$

$$-\frac{1}{2m-1}.\frac{e^2}{1-e^2}.\frac{x\sqrt{(1-x^2)}}{R^{2m-1}}.$$

Hence, it is clear that $\int d\theta.R^{\pm(2m+1)}$ depends on $\int d\theta.R^{\pm(2m-1)}$, and $\int d\theta.R^{\pm(2m-3)}$; similarly, $\int d\theta.R^{\pm(2m-1)}$ depends on $\int d\theta.R^{\pm(2m-3)}$, $\int d\theta.R^{\pm(2m-5)}$, &c. consequently, $\int d\theta.R^{\pm(2m+1)}$ depends on $\int d\theta.R$ and $\frac{\int d\theta}{R}$.

Examples;

1. Let $2m+1=3 \therefore 2m=2$,

$$\therefore\int\frac{d\theta}{R^3}=\frac{1}{1-e^2}.\int R\,d\theta-\frac{e^2}{1-e^2}\,x\,\frac{\sqrt{(1-x^2)}}{R} \qquad (e)$$

$$=\frac{1}{1-e^2}f-\frac{e^2}{1-e^2}\,x\sqrt{\left(\frac{1-x^2}{1-e^2x^2}\right)},$$

when $x=1$, $\int\frac{d\theta}{R^3}=\frac{1}{1-e^2}.f(1)$.

2. Let $2m+1=5$ $\therefore$ $2m=4$

$$\therefore \int\frac{d\theta}{R^5} = \frac{2}{3}\cdot\frac{2-e^2}{1-e^2}\cdot\int\frac{d\theta}{R^3} - \frac{1}{3(1-e^2)}\cdot\int\frac{d\theta}{R} - \frac{e^2}{3(1-e^2)}\cdot x\,\frac{\sqrt{(1-x^2)}}{R^3}$$

$$= \frac{2}{3}\cdot\frac{2-e^2}{(1-e^2)^2}f - \frac{1}{3\cdot(1-e^2)}\cdot\int\frac{d\theta}{R} - \frac{2}{3}\cdot\frac{2-e^2}{(1-e^2)^2}\,e^2\cdot\frac{x\sqrt{(1-x^2)}}{R}$$

$$-\frac{e^2}{3\cdot(1-e^2)}\cdot\frac{x\sqrt{(1-x^2)}}{R^3};$$

if $x=1$,

$$\int\frac{d\theta}{R^5} = \frac{2}{3}\cdot\frac{2-e^2}{(1-e^2)^2}\cdot f(1) - \frac{1}{3\cdot(1-e^2)}\cdot\int\frac{d\theta}{R}. \qquad (f)$$

This is as commodious a form as any for computation, but it may easily be changed into others; thus, since $\int\frac{d\theta}{R} = f - e\,.\,\frac{df}{de}$,

$\int\frac{d\theta}{R^5} = \frac{3-e^2}{3\cdot(1-e^2)^2}\cdot f(1) + \frac{e}{3\cdot(1-e^2)}\cdot\frac{df}{de}$; (integral taken from $x=0$ to $x=1$);

or, since $\int\frac{d\theta}{R} = \int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}} = \frac{1+e'}{2},\int\frac{du'}{\sqrt{(1-u'^2)(1-e'^2u'^2)}}$

$= \frac{2f'(1)}{1-e'} - f(1)\,\frac{1+e'}{1-e'}$, by equation (a), page 240.

$$\therefore \int\frac{d\theta}{R^5} = \frac{5+3e'^2}{3\cdot(1-e')^2}\cdot\left(\frac{1+e'}{1-e'}\right)^2 f(1) - \frac{2\cdot(1+e')^2}{3(1-e')^3}\cdot f'(1),$$

$$\text{or } = \frac{5+3e'^2}{3\cdot(1-e')^2}\cdot\frac{f(1)}{1-e^2} - \frac{2}{3}\cdot\frac{1}{(1-e^2)(1-e')}\cdot f'(1),$$

the integral being taken from $x=0$ to $x=1$.

The integration of the form $\frac{x^{2n}\,dx}{\sqrt{(1-x^2)}}(1-e^2x^2)^{\frac{\pm 2m+1}{2}}$, depends also on the integration of $R\,\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, and of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$; for, substituting as before, and taking the differential of $x^{2n-1}\sqrt{(1-x^2)}\,.\,R^{2m+1}$, (X), we have

$$dX = \frac{(2m+2n)\,e^2-(2m+1)\;x^{2n-2}\,R^{2m+1}\,.\,d\theta}{e^2} - (2n+2m+1)\,.x^{2n}.\,R^{2m+1}.d\theta$$

$$+(2m+1)\frac{1-e^2}{e^2}.x^{2n-2}.R^{2m-1}.d\theta.$$

Hence, $\int x^{2n}\,.\,R^{2m+1}\,.\,d\theta = \frac{(2n+2m)\,e^2-(2m+1)}{(2n+2m+1)\,e^2}.\int x^{2n-2}.R^{2m+1}\,.\,d\theta$ $\qquad(g)$

$$+\frac{2m+1}{2n+2m+1}\cdot\frac{1-e^2}{e^2}\cdot x^{2n-2}\,R^{2m-1}\,d\theta-\frac{X}{2n+2m+1}.$$

And, since a similar form is true for $\int x^{2n-2}\,.\,R^{2m+1}\,d\theta$, and $\int x^{2n-2}\,.\,R^{2m-1}d\theta$, by continuing the process, we must at length arrive at forms such as $\int x^{\circ}\,.\,R^{2\nu+1}\,d\theta$, $\int x^{\circ}\,.\,R^{2\nu-1}\,d\theta$, which have already been shown to be integrable by $\int R d\theta$, and $\int\frac{d\theta}{R}$; or at forms such as $x^{2\sigma}\,Rd\theta$, $x^{2\sigma}\,.\,\frac{d\theta}{R}$, which are integrable by $\int Rd\theta$, $\int\frac{d\theta}{R}$; for, by preceding form,

$$\int x^{2\sigma}\,Rd\theta=A\int x^{2\sigma-2}\,Rd\theta+B\int x^{2\sigma-2}\frac{d\theta}{R}-x^{2\sigma-1}\sqrt{1-x^2}\,R.$$

Similarly,

$$\int x^{2\sigma-2}\,Rd\theta=A'\int x^{2\sigma-4}\,Rd\theta+B'\int x^{2\sigma-4}\frac{d\theta}{R}-x^{2\sigma-3}\sqrt{1-x^2}\,R,$$

&c.

and $$\frac{x^{2\sigma}\,.\,d\theta}{R}=\frac{x^{2\sigma-2}\,.\,(1-R^2)}{e^2}\cdot\frac{d\theta}{R}=\frac{x^{2\sigma-2}}{e^2}\cdot\frac{d\theta}{R}-\frac{x^{2\sigma-2}}{e^2}\,.\,Rd\theta$$

$$\frac{x^{2\sigma-2}\,.\,d\theta}{e^2\,.\,R}=\frac{x^{2\sigma-4}\,.\,d\theta}{e^4\,.\,R}-\frac{x^{2\sigma-4}}{e^4}\,.\,Rd\theta,$$

&c. &c.

so that, finally, the integrals of $x^{2\sigma}\,.\,Rd\theta.$, $\frac{x^{2\sigma}\,d\theta}{R}$, must be reduced to $\int Rd\theta$, and $\int\frac{d\theta}{R}$.

Hence, the integral of a form such as

$$\left\{A+Bx^2+Cx^4+\text{\&c.}\right\}\frac{(1-e^2\,x^2)^{\pm 2m+1}}{\sqrt{(1-x^2)}}\,.\,dx,\ \text{depends on}\ \int Rd\theta.\text{and}\int\frac{d\theta}{R}.$$

If $2m+1$ be negative, or the integral of $\frac{x^{2n}.\,d\theta}{R^{2m+1}}$ be required, then, substituting in the preceding form, or, by a direct process, taking the differential of $\frac{x^{2n-1}\sqrt{(1-x^2)}}{R^{2m-1}}$ (X), there will result,

$$\int\frac{x^{2n}\,.\,d\theta}{R^{2m+1}}=$$

$$\frac{(e^2-1)(2n-1)+2m-2}{(2m-1)(1-e^2)\,.\,e^2}\cdot\int\frac{x^{2n-2}.\,d\theta}{R^{2m-1}}+\frac{(2n-2m+1)}{(2m-1)(1-e^2)\,.\,e^2}\cdot\int\frac{x^{2n-2}\,d\theta}{R^{2m-3}}-\frac{X}{(2m-1)(1-e^2)};$$

and, consequently, $\int \frac{x^{2n}\, d\theta}{R^{2m+1}}$ finally depends on the integrals of $R d\theta$, and of $\frac{d\theta}{R}$.

The expressions hitherto given, are analytical. By the introduction of geometrical language, there arise forms such as $d\theta\sqrt{(1-e^2 . \sin.^2\theta)}$, $d\theta\sqrt{(1-e^2 \cos.^2\theta)}$, $d\theta\sqrt{\left\{1+\frac{e^2}{1-e^2}, (\cos.\theta)^2\right\}}$, $d\theta\sqrt{(1+m . \cos.\theta)}$, $d\theta\sqrt{(m. \cos.\theta+1)}$, $(\cos.\theta)^n . d\theta\left\{1+m \cos.\theta\right\}^{\frac{\pm 2\nu+1}{2}}$; the integration of which depends on that of $dx\sqrt{\left(\frac{1-e^2 x^2}{1-x^2}\right)}$, and of $\frac{dx}{\sqrt{(1-x^2)(1-e^2 x^2)}}$, as might easily be shewn. I shall, however, omit the proof, and only observe, that this variety of expression, by rendering obscure, or remote, the origin of differential expressions, is rather an inconvenience than a benefit to science.

Before I quit this subject, I wish to shew how, from the preceding integrals and methods, the coefficients in the series $A+B. \cos.\theta+C.\cos. 2\theta+$ &c. the expansion of $(a^2+b^2-2ab . \cos.\theta)^{\frac{2m+1}{2}}$, may be determined and computed.

$$\left\{a^2+b^2-2ab.\cos.\theta\right\}^{\frac{2m+1}{2}}=a^{2m+1}\left\{1+\frac{b^2}{a^2}-\frac{2b}{a}\cos.\theta\right\}^{\frac{2m+1}{2}}$$

$$=(\text{if } \tfrac{b}{a}=e')\ a^{2m+1}\left\{1+e'^2-2e'.\cos.\theta\right\}^{\frac{2m+1}{2}}=A+B.\cos.\theta+C.$$ $\cos. 2\theta+$&c.; consequently, $a^{2m+1}.\int(1+e'^2-2e'.\cos.\theta)^{\frac{2m+1}{2}}d\theta=$ $A\theta+B \sin.\theta+\frac{C . \sin. 2\theta}{2}+$ &c. Let $\theta=\pi$

$\therefore A\pi=a^{2m+1}\int(1+e'^2-2e'.\cos.\theta)^{\frac{2m+1}{2}}d\theta$ (when θ is put $=\pi$).

Now, $1+e'^2-2e'.\cos.\theta=1+e'^2-2e'\left\{2\left(\cos.\frac{\theta}{2}\right)^2-1\right\}$

$$=(1+e')^2\left\{1-\frac{4e'}{(1+e')^2}\left(\cos.\frac{\theta}{2}\right)^2\right\}, \text{ let } \frac{4e'}{(1+e')^2}=e^2, \cos.\frac{\theta}{2}=x$$

$$\therefore 1+e'^2-2e'.\cos.\theta=(1+e')^2\left\{1-e^2 x^2\right\},$$

and $\therefore A\pi = a^{2m+1} . (1+e')^{2m+1} . \int (1-e^2 x^2)^{\frac{2m+1}{2}} . \frac{2dx}{\sqrt{(1-x^2)}}$

$= a^{2m+1} . (1+e')^{2m+1} . \int R^{2m+1} d\theta$, which, by what has preceded, can always be determined by means of $\int R d\theta, \int \frac{d\theta}{R}$. To determine B,

$a^{2m+1} . (1+e')^{2m+1} . R^{2m+1} . \cos. \theta = A . \cos. \theta + B . (\cos. \theta)^2 +$ &c.

$\therefore a^{2m+1} . (1+e')^{2m+1} . \int R^{2m+1} . \cos. \theta . d\theta = A . \sin. \theta + \frac{B \sin. 2\theta}{4} + \frac{B\theta}{2} +$ &c.

making $\theta = \pi$, $\sin. \theta$, $\sin. 2\theta$, &c. $= 0$;

consequently, $(a . (1+e'))^{2m+1} . \int R^{2m+1} . \cos. \theta . d\theta . = B \frac{\pi}{2}$; which integral can always be expressed by finite algebraic forms, and the integrals of $R d\theta, \frac{d\theta}{R}$; for, putting $x = \cos. \frac{\theta}{2}$,

$\cos. \theta = 2x^2 - 1$, we have $R^{2m+1} . \cos. \theta . d\theta = (2x^2 - 1) . R^{2m+1} d\theta$

$= 2x^2 . R^{2m+1} . d\theta - R^{2m+1} d\theta$,

and, generally, to determine the coefficient (N) belonging to $\cos. n\theta$,

$\{a . (1+e')\}^{2m+1} . R^{2m+1} \cos. n\theta = A \cos. n\theta + B \cos. n\theta . \cos. \theta +$ &c. $+ N . \{\cos. n\theta\}^2 + N' . \cos. n\theta . \cos. (n+1)\theta +$ &c.

multiply each side by $d\theta$, and integrate, making $\theta = \pi$; then, since

$\int \cos. m\theta . \cos. (m \pm p)\theta d\theta = \int \frac{1}{2} \cos. (2m \pm p)\theta d\theta + \int \frac{1}{2} \cos. p\theta d\theta$

$$= \frac{1}{2} \sin. \frac{(2m \pm p)\theta}{2m \pm p} + \frac{1}{2} \frac{\sin. p\theta}{p}$$

$$= \frac{1}{2} . \sin. \frac{(2m \pm p)\pi}{2m \pm p} + \frac{1}{2} \frac{\sin. p\pi}{p} = 0,$$ and, since

$\int N . (\cos. n\theta)^2 d\theta = \frac{1}{2} \int N d\theta . (\cos. 2n\theta + 1) = \frac{1}{2 . 2n} . N \sin. 2n\theta + \frac{N\theta}{2}$

$= N \frac{\pi}{2} = \{a . (1+e')\}^{2m+1} . \int R^{2m+1} d\theta . \cos. n\theta$, and the integral of $R^{2m+1} \cos. n\theta . d\theta$ can always be determined in terms consisting of finite algebraic quantities, and of the integrals $\int R d\theta, \int \frac{d\theta}{R}$; for,

if x be cosine of $\frac{\theta}{2}$, $\cos. n\theta =$
$\frac{1}{2}\left\{(2x^2-1)^n - n\,.\,(2x^2-1)^{n-2} + \frac{n\,.\,n-3}{1\,.\,2}\,.\,(2x^2-1)^{n-4} - \&c.\right\}$

Example. Suppose $2m+1=-3$,

$\therefore A = \frac{1}{\pi\, a^3\,.\,(1+e')^3}\,.\int\frac{d\theta}{R^3}$. Now, by form ($e$), page 262, f being integral of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, $\int\frac{d\theta}{R^3} = \frac{2f(1)}{1-e^2}\ (x=1)$;

consequently, $A = \frac{2}{\pi\,(a^3\,(1+e')^3)}\,.\,\frac{f(1)}{1-e^2} = \frac{1}{a^3\,.\,(1+e')\,(1-e')^2}\,.\,\frac{2f(1)}{\pi}$

$$= \frac{2f(1)}{(a+b)\,(a-b)^2\,.\,\pi};$$

and, to compute this quantity, the series (6), page 244, or the series (9), page 249, may be used; that is, if $\frac{a-b}{a+b}$ be $> \sqrt{\frac{1}{2}}$, it is most commodious to employ series (6), if $< \sqrt{\frac{1}{2}}$, it is most commodious to employ series (9).

To determine B,

$$\frac{B\pi}{2} = \frac{1}{a^3\,.\,(1+e')^3}\,.\int\frac{\cos.\theta\,.\,d\theta}{R^3},\ \text{but}\ \frac{\cos.\theta\,.\,d\theta}{R^3} = \frac{(2x^2-1)\,2dx}{\sqrt{(1-x^2)}\,.\,R^3},$$

x being the cosine of $\frac{\theta}{2}$.

Now, by form (g), page 263,

$$\int\frac{x^2\,dx}{\sqrt{(1-x^2)}\,.\,R^3} = \frac{1}{e^2.(1-e^2)}.f(1) - \frac{1}{e^2}.\int\frac{dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)},$$

$$\text{and}\int\frac{dx}{\sqrt{(1-x^2)}.\,R^3} = \frac{f(1)}{1-e^2};$$

hence,

$$\int\frac{\cos.\theta\,d\theta}{R^3} = \frac{2\,(2-e^2)}{e^2.\,1-e^2}f(1) - \frac{4}{e^2}\int\frac{dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)}.$$

$$= \frac{(1+e'^2)\,(1+e')^2}{e'.\,(1-e')^2}.f(1) - \frac{(1+e')^2}{e'}\int\frac{dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)}.$$

Hence, calling $\int\frac{dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)}$, (from $x=0$ to $x=1$) F(1),

we have $B = \frac{a^2+b^2}{ab\,.\,(a-b)^2\,(a+b)}\,.\,\frac{2f(1)}{\pi} - \frac{1}{ab\,.\,(a+b)}\,.\,2F(1)$,

if $e' = \frac{b}{a}$ be $\angle \sqrt{2} - 1$, compute $f(1)$ from series (6), and $F(1)$ from the series $(1+e')(1+e'')(1+e''') \ldots (1+\epsilon) \cdot \frac{\pi}{2}$ $\left(\frac{P\pi}{2}\right)$, to which it is equal,

if $e' = \frac{b}{a}$ be $\measuredangle \sqrt{2} - 1$, compute $f(1)$ from series (9), and $F(1)$ from the series $(1+{}'b)(1+{}''b)(1+{}'''b) \ldots (1+\beta) \cdot \frac{\text{hyp. log.} \frac{4}{{}^{(m)}b}}{2^m}$ $\left({}'P \cdot \frac{\text{h. l.} \frac{4}{{}^{(m)}b}}{2^m}\right)$, to which it is equal.

For the purposes of computation, the foregoing expression for B is, I believe, as simple as any that can be proposed. It is easy, however, by means of the preceding forms, to express it differently; thus, $\int \frac{dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)}$ (from $x=0$ to $x=1$)

$= -\frac{1+e'}{1-e'} \cdot f(1) + \frac{1}{1-e'} \cdot 2f'(1).$

Consequently, $\int \frac{\cos.\theta \,.\, d\theta}{R^3}$

$$= \frac{(1+e')^2 \cdot (1+e'^2)}{e'(1-e')^2} \cdot f(1) + \frac{(1+e')^3}{e'(1-e')} \cdot f(1) - \frac{(1+e')^2}{e'(1-e')} \cdot 2f'(1)$$

$$= \frac{2 \cdot (1+e')^2}{e' \cdot (1-e')^2} \cdot f(1) - \frac{(1+e')^2}{e' \cdot (1-e')} \cdot 2f'(1)$$

$$\therefore B = \frac{2}{\pi} \cdot \frac{1}{(a+b)^3} \cdot \int \frac{\cos.\theta \,.\, d\theta}{R^3} = \frac{2a}{b \cdot (a+b)(a-b)^2} \cdot \frac{2f(1)}{\pi} - \frac{2}{b(a^2-b^2)} \cdot 2f'(1)$$

$$\text{or} = \frac{2aA}{b} - \frac{2}{b \cdot (a^2-b^2)} \cdot 2f'(1).$$

Let $2m+1 = -1$;

then, $A\pi = \frac{1}{a \cdot (1+e')} \int \frac{2dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)}$ $(x=1) = \frac{1}{a+b} 2F(1)$,

and $\frac{B\pi}{2} \times a(1+e') = \int \frac{(2x^2-1) \,.\, 2dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)}$

$$= \frac{4}{e^2} \int \frac{dx}{\sqrt{1-x^2}\,(1-e^2x^2)} - \frac{4}{e^2} \int dx \sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)} - 2\int \frac{dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)}$$

$$= \frac{2(2-e^2)}{e^2} \int \frac{dx}{\sqrt{(1-x^2)}\,(1-e^2x^2)} - \frac{4}{e^2} \cdot f = \frac{1+e'^2}{e'} \cdot F(1) - \frac{(1+e')^2}{e'} f(1).$$

Consequently, $B = \frac{a^2+b^2}{ab \cdot a+b} \cdot \frac{2F(1)}{\pi} - \frac{a+b}{ab} \cdot \frac{2f(1)}{\pi}$;

since $F(1) = (1+e')(1+e'') \ldots \frac{\pi}{2} = P \cdot \frac{\pi}{2}$,

$$A = \frac{1}{a+b} \cdot P \text{ (if } a \text{ be put} = 1) = \frac{P}{1+e'} = (1+e'')(1+e''') \&c.$$

Again, $B = \frac{1+e'^2}{e' \cdot 1+e'} \cdot \frac{2F(1)}{\pi} - \frac{1+e'}{e'} \cdot \frac{2f(1)}{\pi}$;

but $f(1) = \frac{\pi}{2} \cdot (1-eQ) P$,

$$\therefore B = \frac{2P}{(1+e')} \cdot \left(\frac{2Q}{e} - 1\right)$$

$$\text{or}, = \frac{2P}{1+e'} \left\{ \frac{e'}{2} + \frac{e' \cdot e''}{2 \cdot 2} + \frac{e' \cdot e'' \cdot e'''}{2 \cdot 2 \cdot 2} + \&c. \right\},$$

$$\text{or} = (1+e'')(1+e''')(1+e^{\text{iv}}) \&c. \left\{ e' + \frac{e' \cdot e''}{2} + \frac{e' \cdot e'' \cdot e'''}{2 \cdot 2} + \&c. \right\}$$

which agrees with the result given by Mr. IVORY, Edinburgh Transactions, Vol. IV. p. 187.

If, instead of the series used for $F(1)$, $f(1)$, we employ the series $\frac{'P}{2^m} \cdot \text{hyp. log.} \frac{4}{(^m)b}$,

$$1 - b \cdot \left\{ \frac{b}{2} \cdot \frac{1+'b}{2} + \frac{b \cdot 'b \cdot (1+'b)}{2 \cdot 2 \cdot 2} \frac{1+''b}{2} + \&c. \right\} + \frac{b \cdot 'P 'Q}{2^m} \cdot \text{hyp. log.} \frac{4}{(^m)b},$$

we shall obtain expressions for A and B, which, in certain values of b, are more commodious for computation than the preceding expressions.

In like manner, if $2m+1 = -5$,

$$A = \frac{8 \cdot (a^2+b^2)}{3 \cdot (a^2-b^2)^3 \cdot (a-b)\pi} f(1) - \frac{\cdot 2 \cdot F(1)}{3 \cdot (a^2-b^2)^2 \cdot (a+b)},$$

$$B = \frac{a}{b} \cdot \frac{a^4 + 14a^2 b^2 + b^4}{3 \cdot (a^2-b^2)^4 (a+b)} \pi f(1) - \frac{a(a^2+b^2)}{3 \cdot b(a^2-b^2)^2 (a+b)} F(1).$$

Since $N = \frac{2}{\pi} \cdot (a \cdot (1+e'))^{2m+1} \int R^{2m+1} \cdot \cos. n\theta \cdot d\theta$; by what has preceded, N may always be determined by a direct process, and independently of the preceding terms. For the purposes of computation, however, it is commodious to deduce N from the

two preceding coefficients $''N$, $'N$; and the method of deduction nearly the oldest, that of CLAIRAUT,* seems to me the best. It is, in substance, nearly as follows.

$$1+e'^2-2e'.\cos.\theta = 1+e'^2\left(1-\frac{2e'}{1+e'^2}.\cos.\theta\right)$$

$$=1+e'^2(1-c.\cos.\theta)=(1+e'^2)V^2, \text{ putting } c=\frac{2e'}{1+e'^2}, V^2=1-c.\cos.\theta.$$

Hence, $$''N\frac{\pi}{2}=(1+e'^2)^{\frac{2m+1}{2}}\int V^{2m+1}.\cos.(n-2)\theta.d\theta,$$

$$'N\frac{\pi}{2}=(1+e'^2)^{\frac{2m+1}{2}}\int V^{2m+1}.\cos.(n-1)\theta.d\theta,$$

$$N\frac{\pi}{2}=(1+e'^2)^{\frac{2m+1}{2}}\int V^{2m+1}.\cos.n\theta.d\theta;$$

consequently, it is necessary to determine

$\int V^{2m+1}.\cos.n\theta.d\theta\,(F'')$ from $\int V^{2m+1}.\cos.(n-1)\theta\,d\theta.(F')$, and $\int V^{2m+1}.\cos.(n-2)\theta.d\theta.(F)$.

Now, $\frac{1}{2}.\cos.n\theta+\frac{1}{2}.\cos.(n-2)\theta=\cos.(n-1)\theta.\cos.\theta$,

$$\therefore \tfrac{1}{2}dF''+\tfrac{1}{2}dF = V^{2m+1}d\theta.\cos.(n-1)\theta.\cos.\theta$$

$$= V^{2m+1}d\theta.\cos.(n-1)\theta\left(\frac{1-V^2}{c}\right)$$

$$=\frac{dF'}{c}-\frac{\cos.(n-1)d\theta}{c}:V^{2m+3};$$

but, $$d\left\{V^{2m+3}.\sin.(n-1)\theta\right\}=(n-1)\cos.(n-1)\theta.V^{2m+3}d\theta+\frac{(2m+3)c}{4}.\cos.(n-2)\theta.V^{2m+1}-\frac{(2m+3)}{4}c.\cos.n\theta\,V^{2m+1}d\theta.$$

Hence, $$\frac{F''}{2}+\frac{F}{2}=\frac{F'}{c}-\frac{\sin.(n-1)\theta:V^{2m+3}}{(n-1).c}-\frac{(2m+3)}{4(n-1)}F''+\frac{2m+3}{4(n-1)}F,$$

$\therefore$ when $\sin.(n-1)\theta=0$,

$$F''=\frac{4(n-1)F'+(2m+3-2(n-1))cF}{(2n+2m+1)c},$$

or,

$$N=\frac{(4n-4)'N+(2m+5-2n)c''N}{(2m+2n+1)c}.$$

Let $n=2\ \therefore\ 'N=B,\ ''N=2A,\ N=C$,

$$\therefore C=\frac{4B+(2m+1)2cA}{(2m+5)c}=\frac{4B}{(2m+5)c}+\frac{(2m+1)2A}{2m+5}.$$

* *Mém. de l'Academie*, 1754, page 550.

Let $n=3$, then,

$$D=\frac{8C}{(2m+7)c}+\frac{(2m-1)B}{2m+7},$$

$$n=4,\quad E=\frac{12D}{(2m+9)c}+\frac{(2m-3)C}{2m+9}$$

&c.

Since, by the preceding forms, the coefficients A, B, can always be expressed in finite algebraic terms, and in terms involving $\int R d\theta$, $\int\frac{d\theta}{R}$, the problem, that of expanding $(1+e'^2-2e'.\cos.\theta)^{\frac{2m+1}{2}}$, is resolved in its most extensive sense. A and B, however, can be determined most easily, in certain values of the index $\frac{2m+1}{2}$; and mathematicians have therefore given methods for deriving A′, B′, (index $\frac{2m+1}{2}\pm 1$) from A, B, (index $\frac{2m+1}{2}$). A method as eligible as any, depends on a problem similar to the preceding; thus, we may determine A′, B′, from A, B, by deducing the integrals of

$V^{2m-1}d\theta$, $V^{2m-1}.\cos.\theta.d\theta$, from those of

$V^{2m+1}.d\theta$, $V^{2m+1}.\cos.\theta.d\theta$; or, since

$$A\pi=a^{2m+1}.(1+e')^{2m+1}.\ R^{2m+1}.d\theta,$$

$$\frac{B\pi}{2}=a^{2m+1}.(1+e')^{2m+1}.R^{2m+1}.\cos.\theta.d\theta,$$

$$A'\pi=\frac{a^{2m+1}.(1+e')^{2m+1}}{a^2.(1+e')^2}.R^{2m-1}d\theta,$$

$$B'\frac{\pi}{2}=\frac{a^{2m+1}.(1+e')^{2m+1}}{a^2.(1+e')^2}.R^{2m-1}.\cos.\theta.d\theta;$$

and, since $\cos.\theta=2x^2-1$ $\left(x=\cos.\frac{\theta}{2}\right)$.

By substituting, in form *(g)*, page 263, for $2n$, 1, we have

$$\int x^2 R^{2m+1}d\theta=\frac{(2m+2)e^2-(2m+1)}{(2m+3)e^2}.\int R^{2m+1}d\theta+\frac{(2m+1)(1-e^2)}{(2m+3)e^2}.$$

$\int R^{2m-1}d\theta\ (x=1)$

$$\therefore\int(2x^2-1)R^{2m+1}d\theta$$

$$=\frac{(2m+1)e^2-2(2m+1)}{(2m+3)e^2}\int R^{2m+1}d\theta+\frac{2.2m+1}{2m+3}.\frac{1-e^2}{e^2}\int R^{2m-1}d\theta.$$

Consequently,

$$\frac{(2m+3)}{2}e^2\text{B}=(2m+1)(e^2-2)\text{A}+2.(2m+1)(1-e^2)a^2.(1+e')^2\text{A}',$$

$$\text{or } \frac{(2m+3)\,2ab}{(a+b)^2}\,.\,\text{B}=-\frac{(2m+1)\,2\,.\,(a^2+b^2)}{(a+b)^2}\text{A}+2\,.\,(2m+1)(a-b)^2\text{A}',$$

$$\text{or, A}'=\frac{2m+3}{2m+1}\,.\,\frac{ab}{(a^2-b^2)^2}\text{B}+\frac{a^2+b^2}{(a^2-b^2)^2}\text{A}.$$

Again, since $(2x^2-1)\,\text{R}^{2m-1}\,d\theta=\frac{2-e^2}{e^2}\,\text{R}^{2m-1}\,d\theta-\frac{2\text{R}^{2m+1}}{e^2}\,.\,d\theta$ (since $x^2=\frac{1-\text{R}^2}{e^2}$) $=\frac{a^2+b^2}{2ab}\,.\,\text{R}^{2m-1}\,d\theta-\frac{(a+b)^2}{2ab}\,\text{R}^{2m+1}\,d\theta$,

$$\text{B}'(a+b)^2=\frac{a^2+b^2}{2ab}\,.\,(a+b)^2\,2\text{A}'-\frac{(a+b)^2}{2ab}\,2\text{A};$$

and substituting for A′,

$$\text{B}'=\frac{2m+3}{2m+1}\,.\,\frac{a^2+b^2}{(a^2-b^2)^2}\text{B}+\frac{4ab}{(a^2-b^2)^2}\text{A}.$$

The method of deducing the coefficients, by a direct process of integration, from $\text{R}^{2m+1}\,.\,d\theta$, cos. $\theta\,.\,\text{R}^{2m+1}\,d\theta$, &c. differs, when examined, scarcely at all from the method of determining A and B (index $2m+1$) from A′ and B′ (index $(2m+1)-2$); for, in the first method,

$$\int\text{R}^{2m+1}\,d\theta=\alpha\int\text{R}^{2m-1}\,d\theta+\alpha'\int\text{R}^{2m-3}\,d\theta+\text{\&c.}+{}'\alpha\int\text{R}d\theta+{}''\alpha\int\frac{d\theta}{\text{R}}$$

$(x=1)$;

or, by continued reduction,

$=\beta\int\text{R}d\theta+\gamma\int\frac{d\theta}{\text{R}}$; (the Greek characters denoting constant coefficients;)

but, since $\int\text{R}^{2m-1}\,d\theta$, $\int\text{R}^{2m-3}\,d\theta$ &c., $\int\text{R}d\theta$, $\int\frac{d\theta}{\text{R}}$, multiplied into certain constant quantities are respectively equal to the coefficients, A′, A″, A‴, &c. ..... ″A, ′A, the indices being

$$2m-1,\ 2m-3,\ 2m-5,\ \ldots\ldots\ 1,\ -1,$$

it is clear, that by determining $\int\text{R}^{2m+1}\,d\theta$, from $\int\text{R}^{2m-1}\,d\theta$, &c. we, in other words, determine A by A′, A″ ...... ″A, ′A, or, when A′, A″, &c. are reduced to depend on ″A, ′A, by ″A, ′A.

By the method given in the preceding pages, the coefficients are made to depend on the integrals (f, F) of $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, $\frac{dx}{\sqrt{(1-x^2)\,1-e^2x^2)}}$. These integrals, it is necessary to compute; and methods have been given for that purpose for all values of e, and consequently for all values of a and b. If the coefficients are to be determined by deriving A, B, from A′, B′, &c. the best method to be followed, is that given by Mr. IVORY, who determines the coefficients, when the index $2m+1=-1$, in fact, by integrating $\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, $(d\text{F})$, or $\frac{d\theta}{\sqrt{(1-e^2(\sin.\theta)^2}}$, on which A depends, and $dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$ (df), or $d\theta\sqrt{(1-e^2.(\sin.\theta)^2}$, on which B partly depends.*

The author last mentioned, in his valuable Paper inserted in the Edinb. Transactions, first, I believe, applied the method of transforming f, F, into similar integrals f', F′, f'', F″, &c. to the determination of the coefficients A, B, &c.; but the method of transformation belongs to LAGRANGE.† This great mathematician has also solved the problem of the expansion of $(a^2+b^2-2ab.\cos.\theta)^{\frac{2m+1}{2}}$; he determines A and B, when the index $\frac{2m+1}{2}=\frac{1}{2}$, in which case, the series for A and B, with respect to its numerical coefficients, decreases the fastest. But the solution is not general, or, to speak

* B depends on f and F, for

$$\frac{\cos.\theta\,.\,d\theta}{\sqrt{(1-e^2.\left(\cos.\frac{\theta}{2}\right)^2}}=\frac{(2x^2-1)\,2dx}{\sqrt{(1-x^2)\,(1-e^2x^2)}}=\frac{-4}{e^2}.\frac{(1-e^2x^2)\,dx}{\sqrt{(1-x^2)\,(1-e^2x^2)}}$$

$$+2.\frac{(2-e^2)}{e^2}.\frac{dx}{\sqrt{(1-x^2)\,(1-e^2x^2)}}=\frac{-4}{e^2}dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}+\frac{2.(2-e^2)}{e^2}\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}.$$

† I assert this on the authority of LACROIX, having never been able to procure the volume of the Turin Memoirs in which LAGRANGE's method is contained.

more precisely, it would be extremely incommodious to compute A and B from the series ascending by the powers of $\frac{b}{a}$, if b were nearly $= a$.

The method of Lagrange, given in the Berlin Acts for 1781, p. 252, has been followed by Laplace, *Mècanique céleste*, p. 268; and, in that part which relates to the derivation of A, B, from A′, B′, by Lacroix, *Calc. diff.* p. 120; and by Mr. Wallace, Edinb. Transactions, Vol. V. p. 256. But the great difficulty of the problem does not consist in deriving the coefficients from one another, but in computing the value of the first and second; and, for this end, a series that simply expresses the expansion of $(1 - e^2.(\cos.\theta)^2)^{\frac{2m+1}{2}}$ must be inadequate, at least, it cannot be commodious and general.

Clairaut has given a peculiar method for finding A, *Mém. de l'Acad.* 1754, p. 546. Arbogast, *Calcul des Derivations*, p. 359, has given a form for the expansion of $(1 - c.\cos.\theta)^{\frac{2m+1}{2}}$, which agrees with Lacroix's, *Calc. int.* p. 121; but the expansion is inconvenient, for reasons already stated, for the purposes of arithmetical computation.

If we join together certain parts of Legendre's Memoir, we shall obtain a complete solution of the problem of the expansion of $(a^2 + b^2 - 2ab(\cos.\theta)^{\frac{2m+1}{2}}$; for he shows, that E, the integral of $d\theta.\sqrt{(1-e^2(\cos.\theta)^2)}$ may always be resolved into similar integrals E′, E″, or ′E, ″E, or, by continuing the resolution, into E″, E‴, or ″E, ‴E, &c. and, consequently, he shows how E may, in all values of e, be computed; and moreover, he shows that the integral of $(\cos.\theta)^n.(1+\alpha.\cos.\theta)^{\frac{2m+1}{2}}d\theta$, may be always reduced to that of $(1+\alpha.\cos.\theta)^{\frac{2m+1}{2}}d\theta$, and therefore to the integral of

$d\theta\sqrt{(1+\alpha\cos.\theta)}$ and $\frac{d\theta}{\sqrt{(1+\alpha\,.\,\cos.\theta)}}$. Now, the coefficient affecting $\cos.n\theta = a\int\cos.n\theta.d\theta.(1-c.\cos.\theta)^{\frac{2m+1}{2}}$, ($a$ a constant quantity,) and $\cos.n\theta = \frac{1}{2}\left\{(2.\cos.\theta)^n - n.(2.\cos.\theta)^{n-2} + \frac{n.\overline{n-3}}{1\,.\,2}(2\cos.\theta)^{n-4} - \&c.\right\}$

E, E′, &c. LEGENDRE calls ellipses, because the differential of the arc of an ellipse may be represented by an expression as $d\theta\sqrt{(1-e^2(\cos.\theta)^2)}$; but the problem of the expansion of $(a^2+b^2-2ab\cos.\theta)^{\frac{2m+1}{2}}$ requires only the integrals of $\frac{d\theta}{R}$, $d\theta.R$; the determination of which integrals, is totally independent of ellipses, as it is, of all other curves.

That the determination of the coefficients A, B, &c. depended on the integral of $Rd\theta$, which, in a particular application, represents the arc of a conic section, was known to D'ALEMBERT. In the *Récherches sur différens Points importans du Système du Monde*, page 66, he proves that A, B, are respectively equal to $\frac{2M}{c}$, $\frac{2M'}{c}$, c being the semicircumference of a circle, and M, M′, the integrals of $dz(a+b\cos.z)^{\frac{n}{2}}$, and $\cos.z\,.\,dz.(a+b\cos.z)^{\frac{n}{2}}$, when $z=\frac{c}{2}$; which integrals depend, he says, on the rectification of the conic sections; he then adds a remark, which requires some comment and explanation. "Tout se reduit donc à trouver par approximation, "la rectification d'un arc donné dans une section conique; et c'est "à quoi on peut parvenir aisément par différentes méthodes. Mais "je ne m'étendrai pas davantage là-dessus, parce que cette manière "de trouver les inconnues A et A′, me paroit plus curieuse et plus "géometrique que commode pour le calcul." p. 67. D'ALEMBERT, therefore, rejects a method which has since been adopted: the reason, I presume to be this; if he had attempted to find the coefficients by the rectification of the conic sections, he must have reduced the integrals of $dz\,(a+b.\cos.z)^{\frac{n}{2}}$, $\cos.zdz.(a+b\cos.z)^{\frac{n}{2}}$

to a series of terms, as $Z' + Z'' + Z'''$ &c. $+ \int dz\,(a + b\,.\cos. z)^{\frac{1}{2}} + \int \frac{dz}{(a+b\cos. z)^{\frac{1}{2}}}$; and, after this reduction, he must have found the integral of $dz\,(a+b\cos. z)^{\frac{1}{2}}$, which, as he then could only do by resolving it into a series, was a problem not more easy, than the finding of the integral of $dz\,(a+b\cos. z)^{\frac{n}{2}}$ from its immediate resolution into a series; consequently, the reduction of $dz\,(a+b\cos. z)^{\frac{n}{2}}$, into $Z' + Z'' +$ &c. would have been useless and unprofitable labour. Had a certain and easy method of computing $(a + b.\cos. z)^{\frac{1}{2}}$ been known to D'ALEMBERT, he would not have asserted the reduction of $\int (a+b\,.\cos. z)^{\frac{n}{2}}$ into $Z' + Z'' +$ &c. $\int (a+b\cos.)^{\frac{1}{2}}\,dz$, to be a method "plus curieuse que commode."

The integral of $\frac{x^4\,dx}{\sqrt{(1-x^2)}}$ furnishes an easy instance for illustration. Suppose it were necessary to compute it, a value of x being given less than 1; resolve it into

$$\sqrt{(1-x^2)}\left\{Ax^3 + Bx^2 + Cx\right\} + \int E\,.\frac{dx}{\sqrt{(1-x^2)}};$$

then, from such expression, may the integral be easily found, since we have tables that exhibit the value of $\int \frac{dx}{\sqrt{(1-x^2)}}$ for all values of x between 0 and 1; but, if the zeal and ability of former computists had not enabled us, in all cases, to assign the value of $\int \frac{dx}{\sqrt{(1-x^2)}}$, it would be, practically, more easy and convenient, for a single instance, to compute an expression as $\int \frac{x^4\,dx}{\sqrt{(1-x^2)}}$ immediately from

$$\int x^4\,dx\left\{1 - D\,1^{-\frac{1}{2}}x^2 + \underset{c}{D^2}\,1^{-\frac{1}{2}}\,x^4 - \underset{c}{D^3}\,1^{-\frac{1}{2}}\,x^6 + \text{&c.}\right\},$$

$$\text{or } \frac{x^5}{5} - \frac{D\,1^{-\frac{1}{2}}.\,x^7}{7} + \frac{\underset{c}{D^2}\,1^{-\frac{1}{2}}\,x^9}{9} - \text{&c.}$$

than from $\sqrt{(1-x^2)}\left\{Ax^3 + Bx^2 + Cx\right\} + E\int \frac{dx}{\sqrt{(1-x^2)}}$,

since, after this reduction, it would be necessary to compute $E\int \frac{dx}{\sqrt{1-x^2}}$ from $E\left\{x - D\,1^{-\frac{1}{2}}\frac{x^3}{3} + \underset{c}{D^2}\,1^{-\frac{1}{2}}.\frac{x^5}{5} - \&c.\right\}$

These observations are, however, digressive; the problem, the expansion of $(a^2+b^2-2ab\,.\cos\theta)^{\frac{2m+1}{2}}$, is, I conceive, completely resolved in the preceding pages, whatever be the ratio between the radii of the planets' orbits.*

What I have advanced, on a former occasion, concerning the independence of analysis and geometry, is confirmed by the present reasonings and results. $\int dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}$, $\int .dx\sqrt{\left(\frac{e^2x^2-1}{x^2-1}\right)}$, $\int\frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}}$, have been computed, without the introduction of an ellipse, an hyperbola, an oblique cylinder, or a pendulum

* In the case of the new planets Ceres and Pallas, whose mean distances from the sun are nearly equal, the series (8), and the expression $\frac{(1+{}'b)(1+{}''b)(1+{}'''b)\ldots(1+{}^{(m)}b)}{2^m}.\,l.\,\frac{4}{{}^{(m)}b}$ (F), will be very convenient, on account of the rapid convergency of the quantities, ${}'b$, ${}''b$, ${}'''b$, &c.; and, in general, in estimating the disturbing forces of 2 planets, since e' is

$$= \frac{\text{mean distance of nearest planet}}{\text{mean distance of the more remote planet}}, \text{ and}$$

$$b = \frac{1-e'}{1+e'}, \text{ putting } b=e',\ e'=\sqrt{2}-1= .4142, \&c.$$

hence, if e' be greater than .4142, &c. the series of terms ${}'b$, ${}''b$, ${}'''b$, &c. decrease more rapidly than e'', e''', &c. and, consequently, the series (8), page 248, and the series $\frac{(1+{}'b)(1+{}''b)\ldots(1+{}^{(m)}b)}{2^m}.\,l.\,\frac{4}{{}^{(m)}b}$, are to be used in determining the perturbations,

when the planets are Mercury and Venus,	$e'=0.53516076$
Venus and Earth,	$e'=0.72333230$
Venus and Mars,	$e'=0.47472320$
Earth and Mars,	$e'=0.65630030$
Jupiter and Saturn,	$e'=0.54531725$
Saturn and Georgium,	$e'=0.49719638$
Ceres and Pallas,	

vibrating in a circular arc; and, as $\int \frac{dx}{\sqrt{(1-x^4)}}$ might have been computed, without the introduction of the *Lemniscata*.* I have stated the mode by which analysis may derive aid from geometry; the extent of the aid however is, I conceive, very small; remove the circle, ellipse, and parabola, curves whose properties have been the object of so much investigation, and we only create for ourselves unnecessary and circuitous operations, by introducing curves into the discussion of questions purely analytical. For the purposes of classification, however, curves may not be altogether useless.

The correspondence that has been shown, between the artifices of calculation and the properties of geometrical figures, may be thought, perhaps, curious or remarkable; and the reduction of several methods into one is, I presume, practically and scientifically useful. On similar reductions, the perfection of analysis, to a great degree, depends: for, a frequent result of a careful investigation is, the discovery that methods apparently different, because differently expressed, are founded on the same principle and fundamental notion; but, if examination and study thus diminish the seeming bulk of our knowledge, they, at the same time, increase its precision and purity.

* See EULER's Memoir, *Novi Comm.* Tom. VI. p. 37, &c. Likewise, relative to the subject of this Paper, *Novi Comm.* Tom. XII. *Nova Acta,* Tom. VII. 1778.